Synchronisation with group interactions

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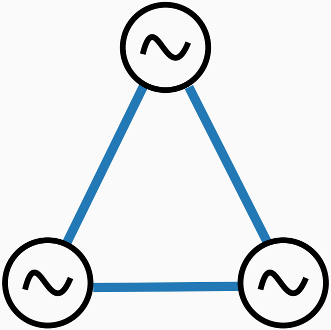
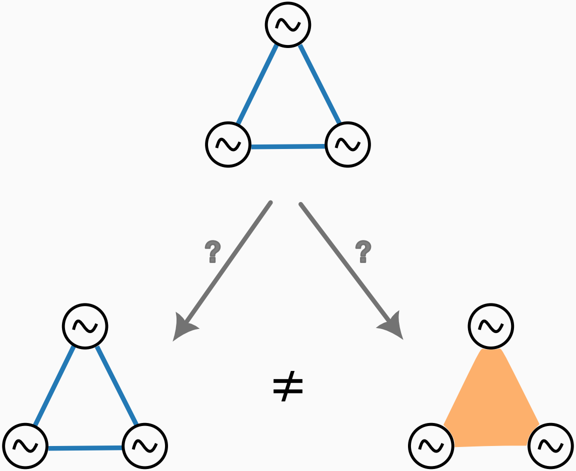
IPAM22, Los Angeles

Work done with G. Cencetti, Y. Zang, and F. Battiston

Networks are great



But.. networks don't encode group interactions



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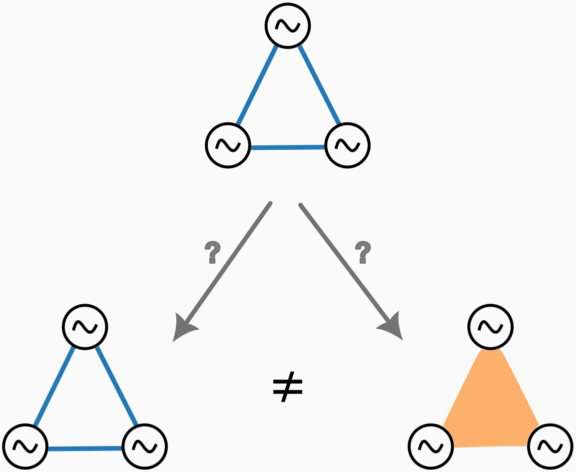


3 papers by 2 authors each

1 paper by 3 authors

Examples

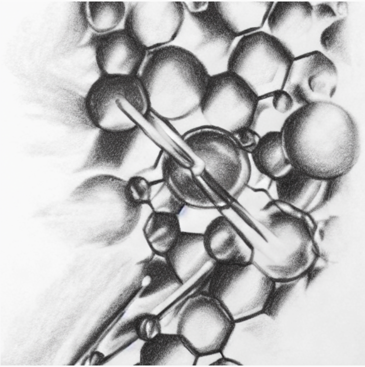
- co-authorship networks

  (A. Patania et al., 2017)
- chemical reactions

  (F. Klimm et al., 2020)
- neuron receiving multiple synaptic inputs

  (Tanaka et al., 2011)
- social dynamics

  (Iacopini et al., 2019)
- ...



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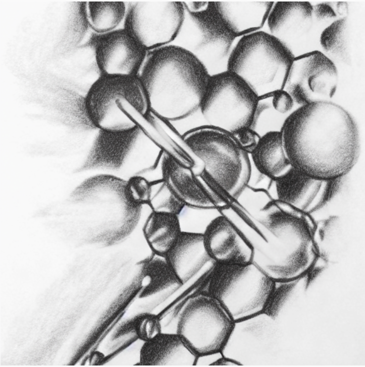


See our big Phys. Rep. review (Battiston et al., 2020). I was the main responsible for the section on synchronisation.

doi.org/10.1016/j.physrep.2020.05.004

Group interactions in sync?

Group interactions can be structural..

or naturally appear from phase reduction.:

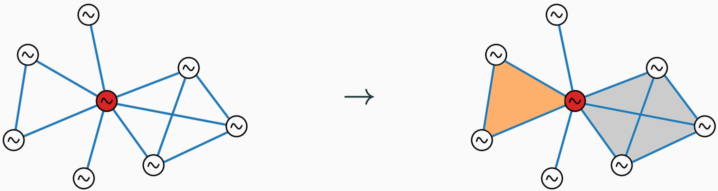
$$\dot{\boldsymbol{x}}_i = F(\boldsymbol{x}) + \epsilon \sum_j \sin(\boldsymbol{x}_j - \boldsymbol{x}_i)$$

becomes

$$\dot{\theta}_i = f(\theta_i) + \epsilon \sum_j \sin(\theta_j - \theta_i) + \epsilon^2 \sin(\theta_k + \theta_j - 2\theta_i) + ...$$

e.g. Leon and Pazo 2019, Gengel et al. 2020

Group interaction can change the dynamics



Group interactions

- can cause explosive sync (Skardal and Arenas, 2019)
- favour chaos (Bick et al., 2016)
- favour clusters (Tanaka, 2011)
- can be infered from data (Kralemann et al., 2011)
- ...

Now: what we did

Let's talk about sync

1. Part 1: Model and the Multiorder Laplacian

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2. Part 2: Do group interactions promote sync?

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Interrupt me with questions!

Part 1:
The multiorder Laplacian

Pairwise Laplacian computes sync stability

In traditional pairwise networks, the Laplacian is a **linear operator** typically used to study full sync.

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We extended the traditional Laplacian to include group interactions $\rightarrow$ multiorder Laplacian.

Lucas et al., 2020

Traditional pairwise Laplacian

Pairwise Kuramoto model:

$$\dot{\theta}_i = \omega + \frac{\gamma_1}{\langle K^{(1)} \rangle} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i)$$

Sync is a solution: $\theta_i(t) = \theta_j(t)$ gives us $\theta(t) = \omega t + cst$.

Traditional pairwise Laplacian

Is it linearly stable? Evolution of an infinitesimal and heterogeneous perturbation around it, $\delta\psi_i(t)$.

$$\delta\dot{\psi}_i = -\sum_{j=1}^{N} \frac{\gamma_1}{\langle K^{(1)} \rangle} L_{ij}^{(1)} \delta\psi_j.$$

with the pairwise Laplacian

$$L_{ij}^{(1)} = K_i \delta_{ij} - A_{ij}$$

Its eigenvalues determine if sync is stable. In particular:

$$\lambda_2 < 0 \quad \text{means stable}$$

Let’s start business

The model

Natural generalisation of the Kuramoto model, with **all possible orders** $d = 1, ..., D$ and **complex structure**:

$$\begin{aligned}
\dot{\theta}_i = \omega &+ \frac{\gamma_1}{\langle K^{(1)} \rangle} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) \\
&+ \frac{\gamma_2}{2! \langle K^{(2)} \rangle} \sum_{j,k=1}^{N} B_{ijk} \sin(\theta_j + \theta_k - 2\theta_i) \\
&+ \frac{\gamma_3}{3! \langle K^{(3)} \rangle} \sum_{j,k,l=1}^{N} C_{ijkl} \sin(\theta_j + \theta_k + \theta_l - 3\theta_i) \\
&+ ... \\
&+ \frac{\gamma_D}{D! \langle K^{(D)} \rangle} \sum_{j_1, ..., j_D = 1}^{N} M_{ij_1, ..., j_D} \sin\left(\sum_{m=1}^{D} \theta_{j_m} - D\,\theta_i \right)
\end{aligned}$$

where D can be at most $N - 1$ (interaction of N oscillators).

Linear stability of synchronised solution

Is sync it linearly stable?

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Is sync it linearly stable?

Introducing the **multi-order Laplacian** $L_{ij}^{(\mathrm{mul})}$, it naturally reduces to just this!

$$\delta\dot{\psi}_i = -\sum_{j=1}^{N} L_{ij}^{(\mathrm{mul})} \delta\psi_j.$$

Stability is only determined by the eigenvalues of this matrix $L_{ij}^{(\mathrm{mul})}$.

Multi-order Laplacian: combine all orders

Multi-order Laplacian: weighted sum of Laplacians of order d

$$L_{ij}^{(\text{mul})} = \frac{\gamma_1}{\langle K^{(1)} \rangle} L_{ij}^{(1)} + \frac{\gamma_2}{\langle K^{(2)} \rangle} L_{ij}^{(2)} + \cdots + \frac{\gamma_D}{\langle K^{(D)} \rangle} L_{ij}^{(D)}$$

Multi-order Laplacian: combine all orders

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Laplacian of order d: natural generalisation of traditional Laplacian

$$L_{ij}^{(d)} = \boldsymbol{d} K_i^{(d)} \delta_{ij} - A_{ij}^{(d)}$$

with, at each order d:

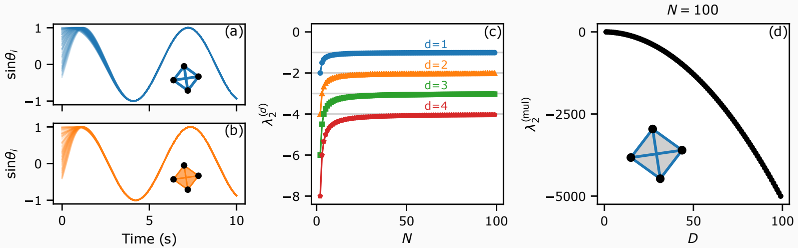
Degree	$K_i^{(d)} = \#$ d-simplices incl. i
Adjacency	$A_{ij}^{(d)} = \#$ d-simplices incl. (i, j)

Ok, let's look at examples

All-to-all attractive

All-to-all at all orders $\rightarrow$ analytical Lyapunov spectrum

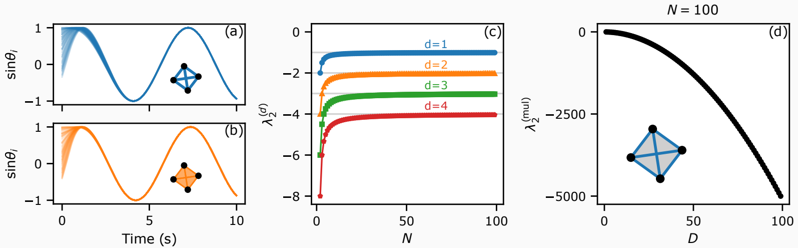
- the larger the order, the "stronger" the interaction: $\lambda_2^{(d)} \propto -d$



All-to-all attractive

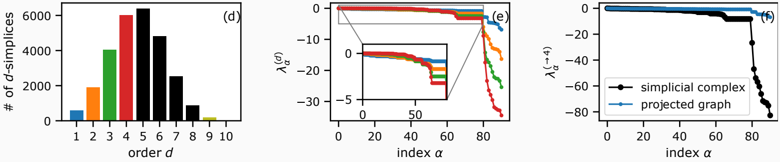
All-to-all at all orders $\rightarrow$ analytical Lyapunov spectrum

- the larger the order, the "stronger" the interaction: $\lambda_2^{(d)} \propto -d$
- including higher orders makes sync more stable



Macaque brain connectome

The multi-order Laplacian can be used to compute the stability of synchronisation in real datasets:



Summary 1

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- The model: phase oscillators with group interactions
- The framework: we **generalised the pairwise Laplacian** to account for group interactions **of any size**
- Group interactions affect structure and hence influence the stability of sync
- Group interactions seem to make sync **more stable**

Case study 2:
do group interactions promote sync?

Open questions

1. **Do group interactions promote sync?**
   examples found in previous studies.
   physically plausible: information travels faster
2. **Does the choice of representation matter?**
   hypergraphs or simplicial complexes:
   no big difference in previous studies

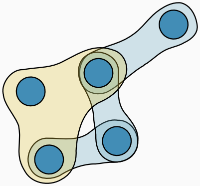
Representation: hypergraphs or simplicial complexes

hypergraph

most general

just a list of hyperedges (set of nodes)

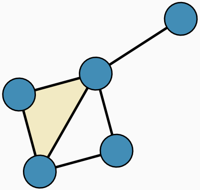
e.g.: [[1,2], [2,3], [3,4], [2, 4, 5]]



simplicial complex

hypergraph with inclusion condition

add [[2, 4], [4, 5], [2,5]] to close the triangle



So far, choice often based on technical convenience.

Battiston et al., 2020

Let's use the multiorder Laplacian

Model: constrained total coupling

Same model, with size up to 2

$$\dot{\theta}_i = \omega + \frac{\gamma_1}{\langle k^{(1)} \rangle} \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i) + \frac{\gamma_2}{2! \langle k^{(2)} \rangle} \sum_{j,k=1}^{n} B_{ijk} \frac{1}{2} \sin(\theta_j + \theta_k - 2\theta_i)$$

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with the constraint

$$\gamma_1 = 1 - \alpha \qquad \gamma_2 = \alpha \qquad \alpha \in [0, 1]$$

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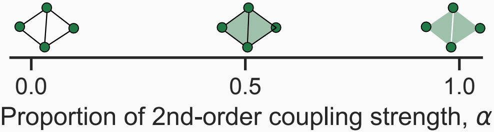
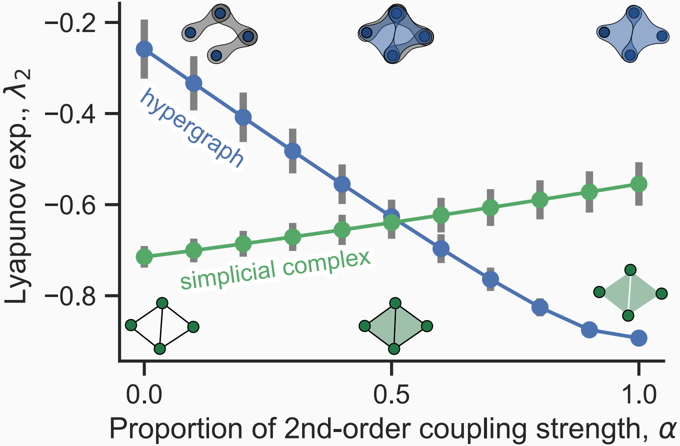


Zhang, Lucas and Battiston, 2022

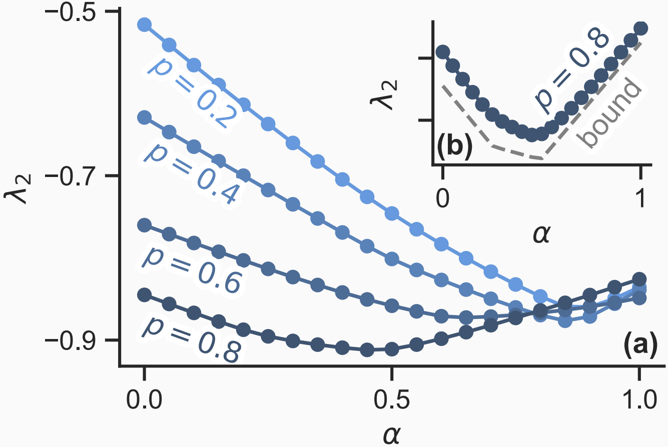
Fixing the total coupling: compare edges and triangles fairly

Simplicial complexes impede sync...

... but random hypergraphs improve it!

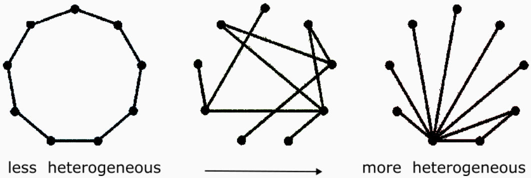


For some parameters, optimum is mixed



p: probability of linking two nodes

Analytical explanation for SCs: rich-gets-richer



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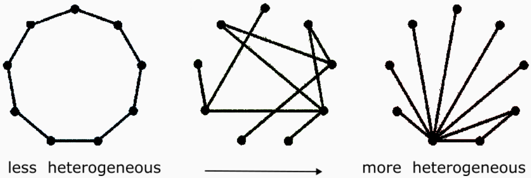

In pairwise (−) networks, we know:

$$\text{degree heterogeneity} \nearrow \quad \rightarrow \quad \text{sync stability} \searrow$$

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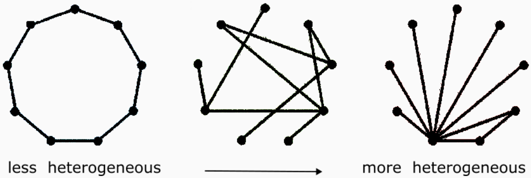

In pairwise ($-$) networks, we know:

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With triangles (▲), we showed:

$$\text{degree heterogeneity } (\blacktriangle) > \text{degree heterogeneity } (-)$$

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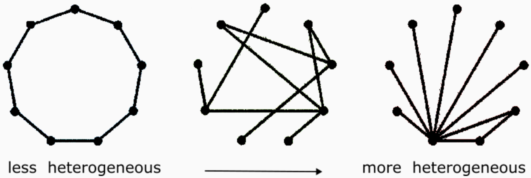

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So that

$$\blacktriangle \text{ strength} \nearrow \quad \rightarrow \quad \text{tot. deg. heterogen.} \nearrow \quad \rightarrow \quad \text{sync stability} \searrow$$

Summary 2

- Group interactions improve sync in random hypergraphs but impede it in simplicial complexes
- Choice of representation affects **degree heterogeneity**
- **The choice of representation is important!**

Python library: XGI

compleX Group Interactions: provides data structures and algorithms for modeling and analyzing complex systems with group (higher-order) interactions.

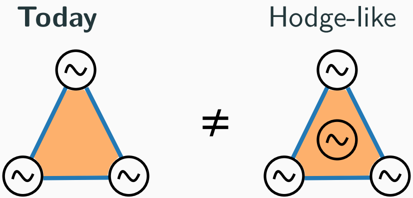
- Github: https://github.com/ComplexGroupInteractions/xgi
- Docs: https://xgi.readthedocs.io/
- Tutorials: https://github.com/ComplexGroupInteractions/xgi/tree/main/tutorials

Take home

- Group interactions **can change the** dynamics
- The **multiorder Laplacian** is a extension of the traditional Laplacian
- Group interactions **do not** always promote sync
- The **choice of representation** actually matters

Broadening the discussion

Two parallel lines of research:



The analytical tools are different:

e.g., we cannot use the Hodge Laplacian.

Direct experimental data?

Less datasets with direct measurements of group interactions than pairwise

- https://github.com/ComplexGroupInteractions/xgi-data
- https://www.cs.cornell.edu/ arb/data/

Direct experimental data?

Inferring from node time series or pairwise interactions

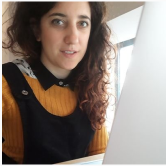
- *Reconstructing phase dynamics of oscillator networks*, Kralemann et al., 2011
- *Principled inference of hyperedges and overlapping communities in hypergraphs*, Contisciani et al., 2022
- *Hypergraph reconstruction from network data*, Young et al., 2021

Final thoughts

- Comparing these models with actual structure and dynamics from experiments?
- Influence of coupling functions?

Work done with

Giulia Cencetti



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Fede Battiston

Thank you for your attention!

Any questions?

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CENTAI

References

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- Multiorder Laplacian for synchronization in higher-order networks. Lucas M., Cencetti G. and Battiston F., 2020. *Phys. Rev. Res.*, 2(3), p.033410.
- Do higher-order interactions promote synchronization? Zhang Y.*, Lucas M.* and Battiston F., 2022. arXiv:2203.03060.