Multilayer networks

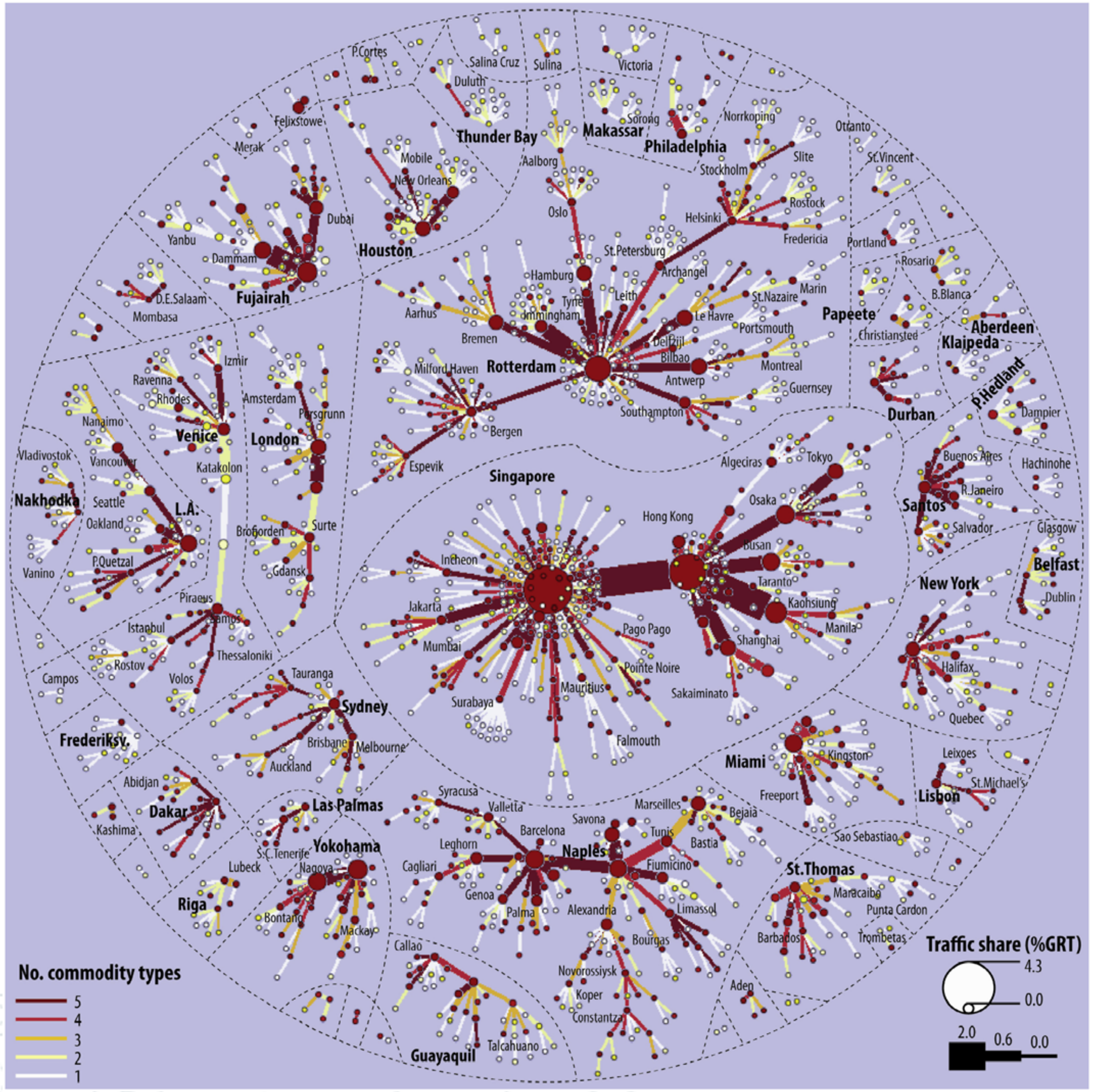
CENTAI

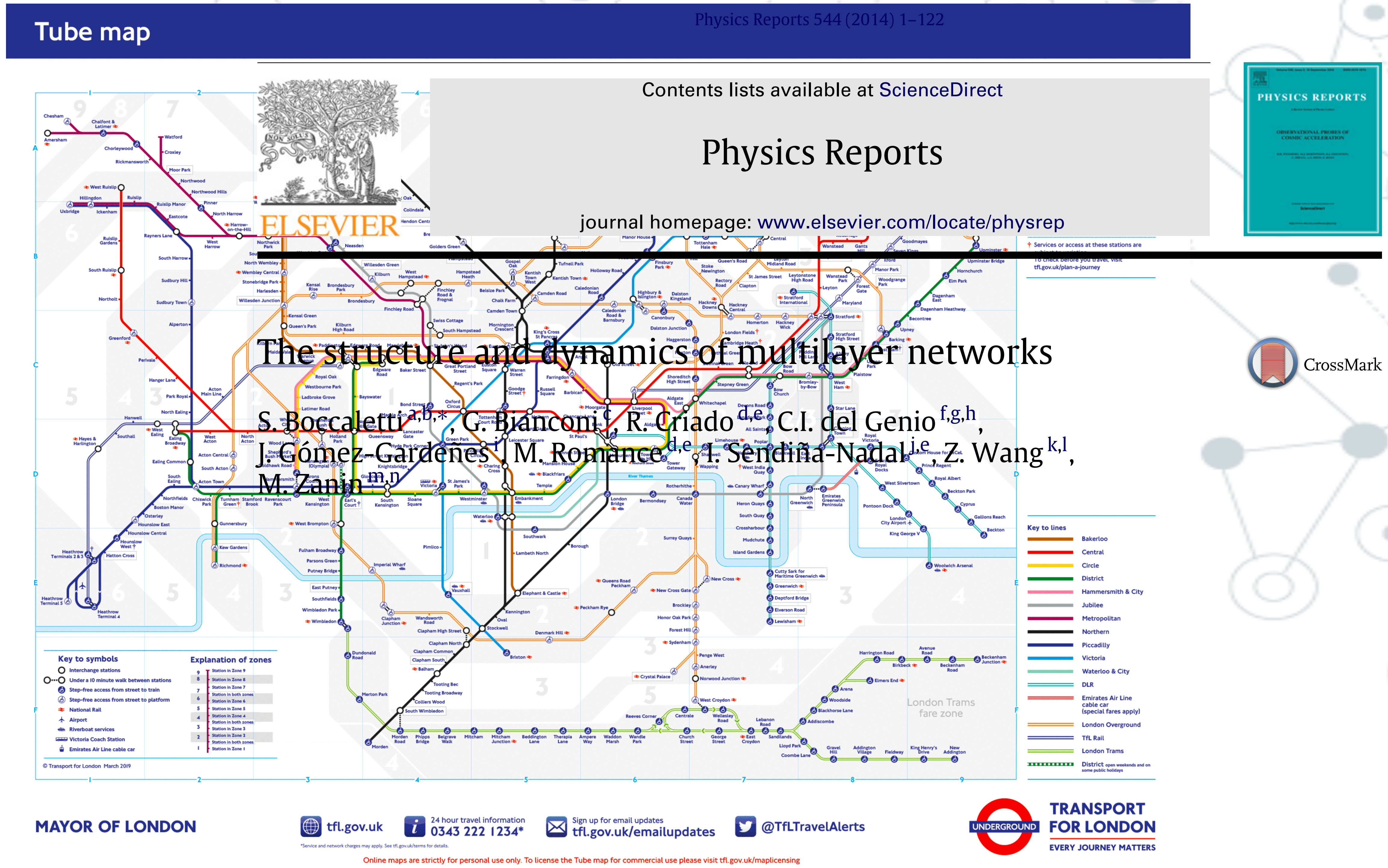
ISI Foundation

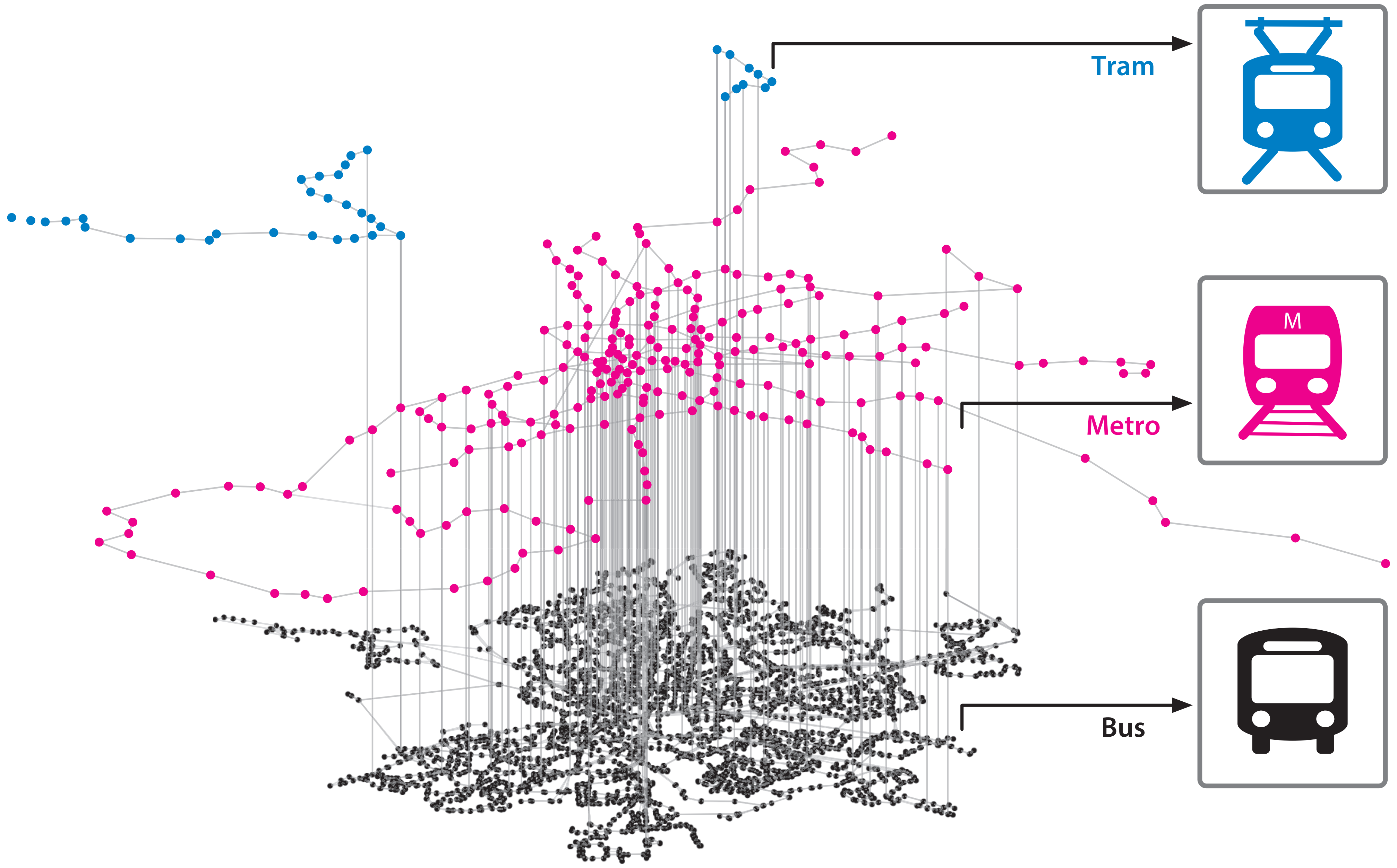
Complexity in Social Systems
AA 2023/2024
Maxime Lucas
Lorenzo Dall'Amico

Multilayer Networks

Examples







Multilayer Networks

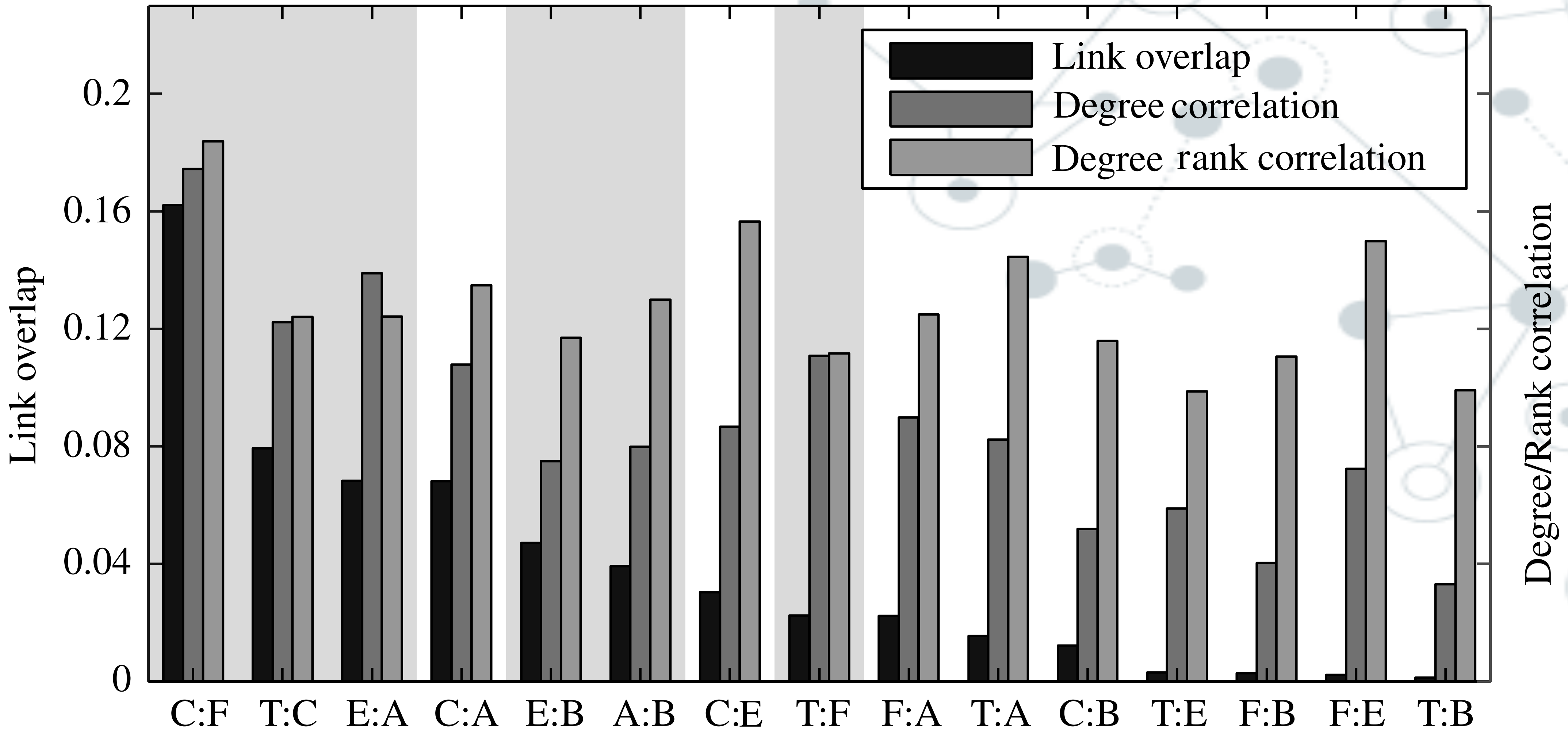
Examples

S. Boccaletti et al. / Physics Reports 544 (2014) 1–122

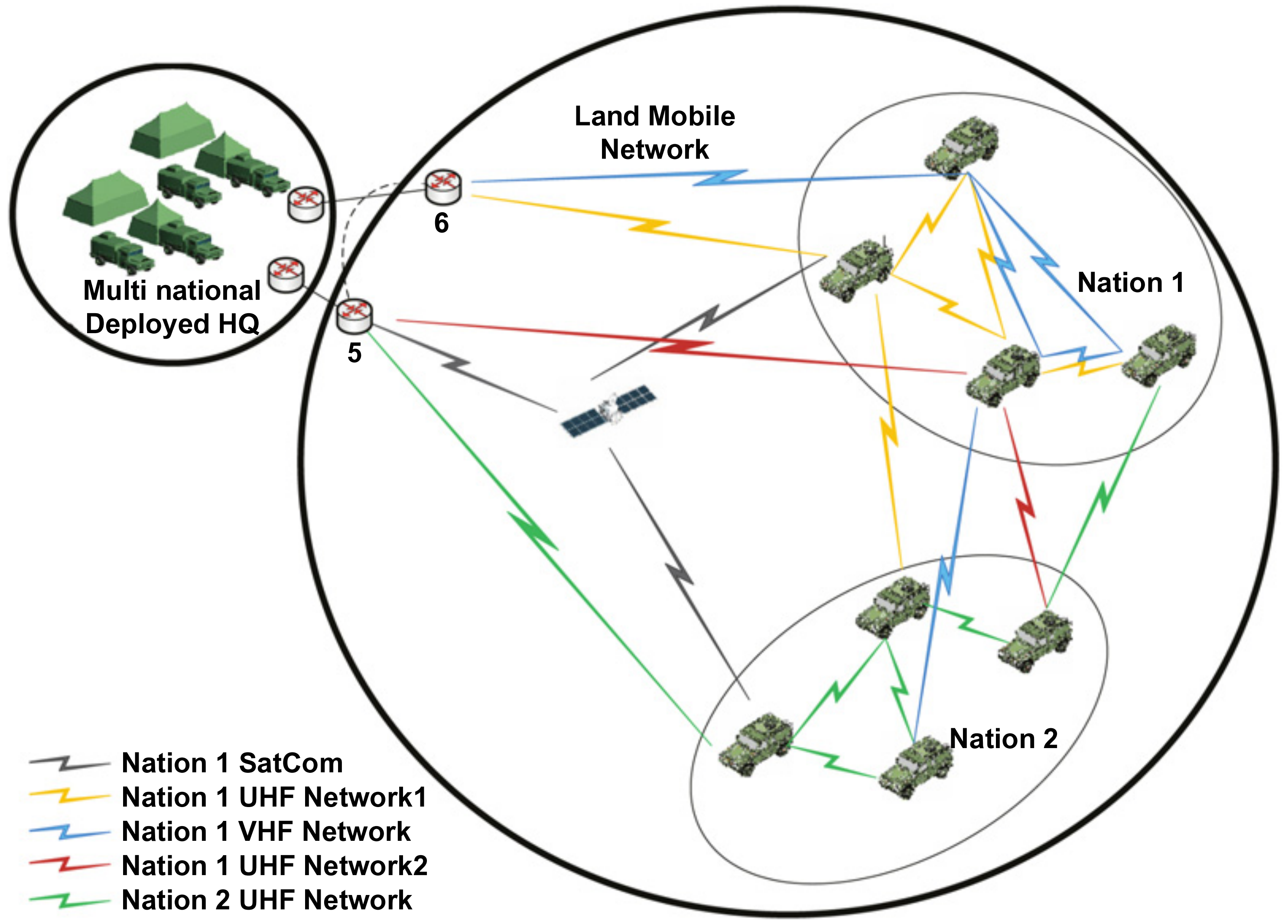
Table 6
Resume of the main application topics, and related references.

Field	Topic	References
Resume of topics and references		
Social	Online communities	Pardus: [63,419–422] Netflix: [423,424] Flickr: [66,88,425] Facebook: [68,426–428] Youtube: [429] Other online communities: [54,89,430] Merging multiple communities: [122,123, 431,432]
	Internet	[109,110,433]
	Citation networks	DBLP: [31,33,434–439]
	Other social networks	Scottish Community Alliance: [440] Politics: [68,441] Terrorism: [23] Bible: [442] Mobile communication: [443]
Technical	Interdependent systems	Power grids: [25,81,444] Space networks: [445]
	Transportation systems	Multimodal: [149,184] Cargo ships: [446] Air transport: [16,78]
	Other technical networks	Warfare: [447]
Economy	Trade networks	International Trade Network: [70,71,448] Maritime flows: [449]
	Interbank market	[450]
	Organizational networks	[451–453]
Other applications	Biomedicine	[454–459]
	Climate	[24,460]
	Ecology	[64,461]
	Psychology	[462]

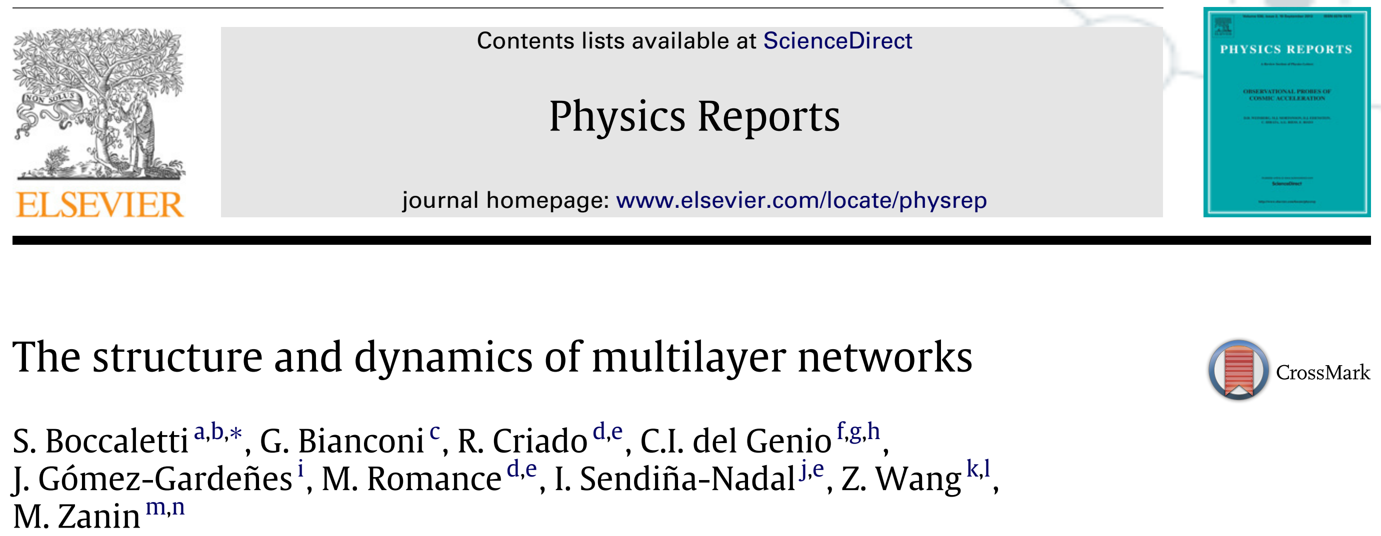
Overlap in Pardus



Zachary Karate Club Club



Multilayer Networks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ELSEVIER

The structure and dynamics of multilayer networks

S. Boccaletti[a,b,*], G. Bianconi[c], R. Criado[d,e], C.I. del Genio[f,g,h], J. Gómez-Gardeñes[i], M. Romance[d,e], I. Sendiña-Nadal[j,e], Z. Wang[k,l], M. Zanin[m,n]

Formal definition

$$\mathcal{M} = (\mathcal{G}, \mathcal{C}) \text{ where } \mathcal{G} = \{G_\alpha;\ \alpha \in \{1, \dots, M\}\}$$

$$G_\alpha = (X_\alpha, E_\alpha)$$ Intralayer

$$\mathcal{C} = \{E_{\alpha\beta} \subseteq X_\alpha \times X_\beta;\ \alpha, \beta \in \{1, \dots, M\},\ \alpha \neq \beta\}$$ Interlayer

$$X_\alpha = \{x_1^\alpha, \dots, x_{N_\alpha}^\alpha\}$$ Nodes in layer

$$a_{ij}^\alpha = \begin{cases} 1 & \text{if } (x_i^\alpha, x_j^\alpha) \in E_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

Interlayer connections

$$a_{ij}^{\alpha\beta} = \begin{cases} 1 & \text{if } (x_i^\alpha, x_j^\beta) \in E_{\alpha\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Projected graph

$$proj(\mathcal{M}) = (X_\mathcal{M}, E_\mathcal{M})$$

$$X_\mathcal{M} = \bigcup_{\alpha=1}^{M} X_\alpha, \qquad E_\mathcal{M} = \left(\bigcup_{\alpha=1}^{M} E_\alpha\right) \bigcup \left(\bigcup_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{M} E_{\alpha\beta}\right).$$

Multiplex network

$$X_1 = X_2 = \cdots = X_M = X \qquad E_{\alpha\beta} = \{(x, x);\ x \in X\}$$

Mono-layer multiplex representation network

$$\tilde{\mathcal{M}} = (\tilde{X}, \tilde{E}) \quad \tilde{X} = \bigsqcup_{1\leq\alpha\leq M} X_\alpha = \{x^\alpha;\ x \in X_\alpha\}$$

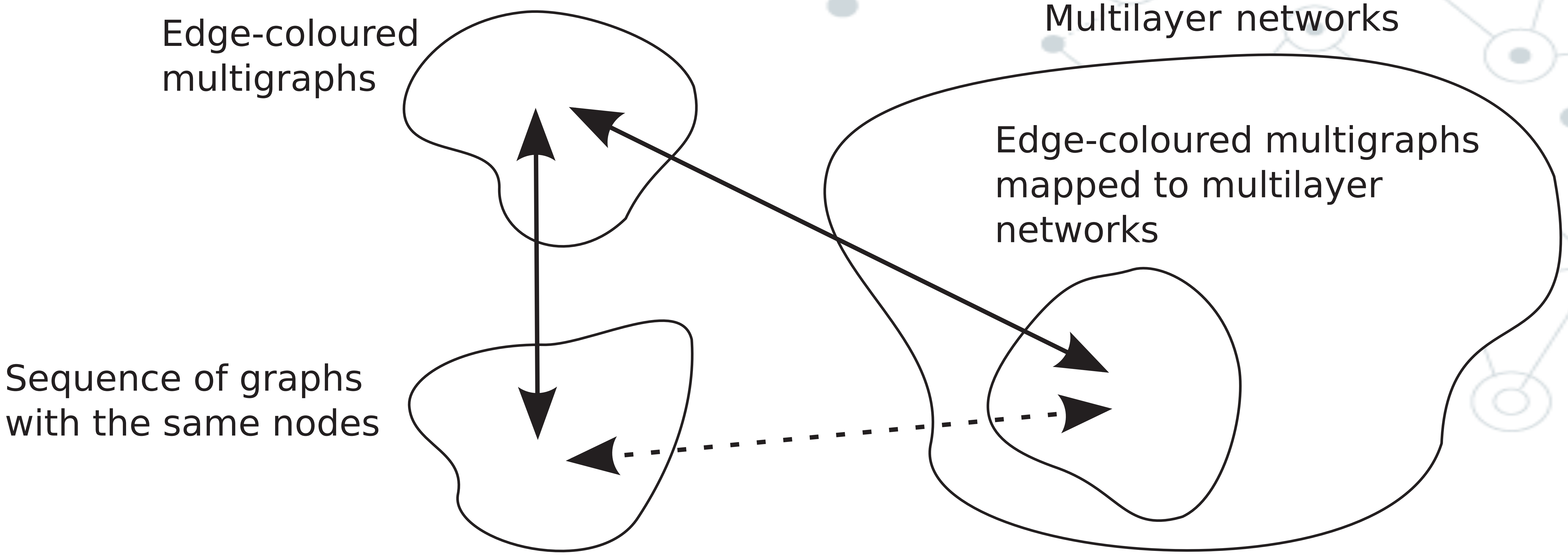
edges $$\left(\bigcup_{\alpha=1}^{M} \{(x_i^\alpha, x_j^\alpha);\ (x_i^\alpha, x_j^\alpha) \in E_\alpha\}\right) \bigcup \left(\bigcup_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{M} \left\{\left(x_i^\alpha, x_i^\beta\right);\ x_i \in X\right\}\right).$$

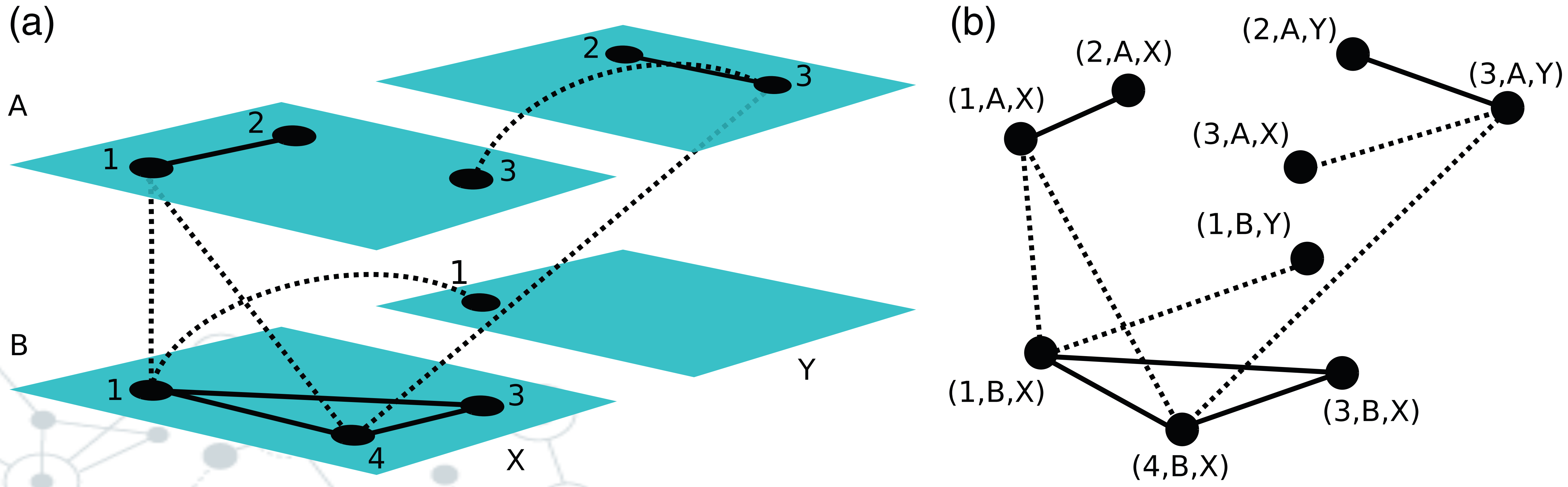
$$\tilde{A} = \left(\begin{array}{c|c|c|c} A_1 & I_N & \cdots & I_N \\ \hline I_N & A_2 & \cdots & I_N \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline I_N & I_N & \cdots & A_M \end{array}\right) \in \mathbb{R}^{NM \times NM},$$

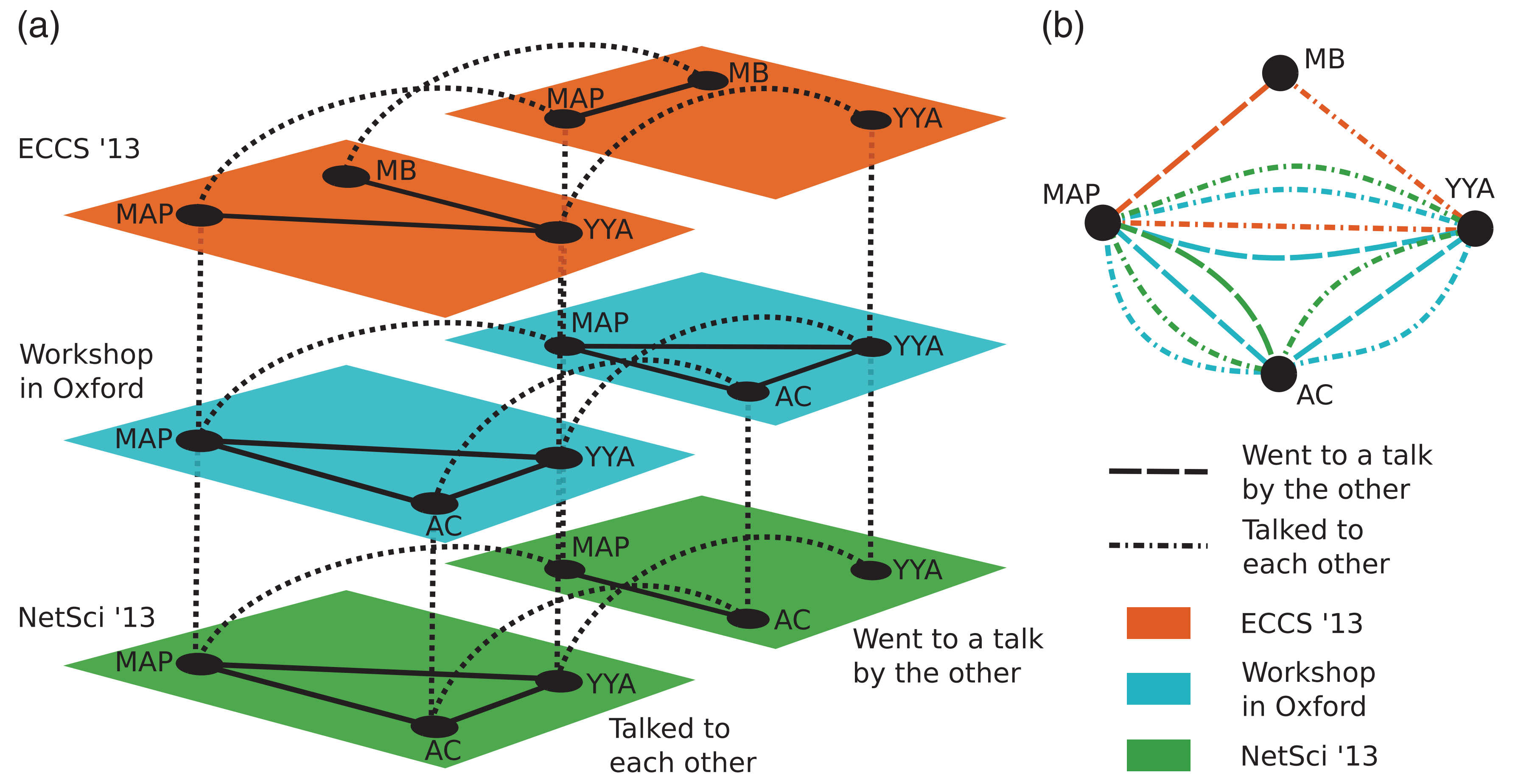
Multilayer Networks

Relations to other extended networks

1. Multiplex networks
2. Temporal networks
3. Interacting networks







Multilayer Networks

Observables

Degree vector

$$\mathbf{k}_i = (k_i^{[1]}, \ldots, k_i^{[M]}),$$

Overlapping degree

$$o_i = \sum_{\alpha=1}^{M} k_i^{[\alpha]},$$

Degree entropy

$$H_i = -\sum_{\alpha=1}^{L} \frac{k_i^{\alpha}}{o_i} \ln\left(\frac{k_i^{\alpha}}{o_i}\right),$$

Eigenvector centrality

$$\mathbf{c}_i = (c_i^{[1]}, \ldots, c_i^{[M]}) \in \mathbb{R}^M,$$

Independent layer eig-centrality

$$C = \left(\; \mathbf{c}_1^T \mid \mathbf{c}_2^T \mid \ldots \mid \mathbf{c}_M^T \; \right) \in \mathbb{R}^{N\times M}.$$

Uniform eigenvector-like centrality

$$\widetilde{A} = \sum_{\alpha=1}^{M} (A^{[\alpha]})^{\mathrm{T}},$$

local heterogeneous eigenvector-like centrality

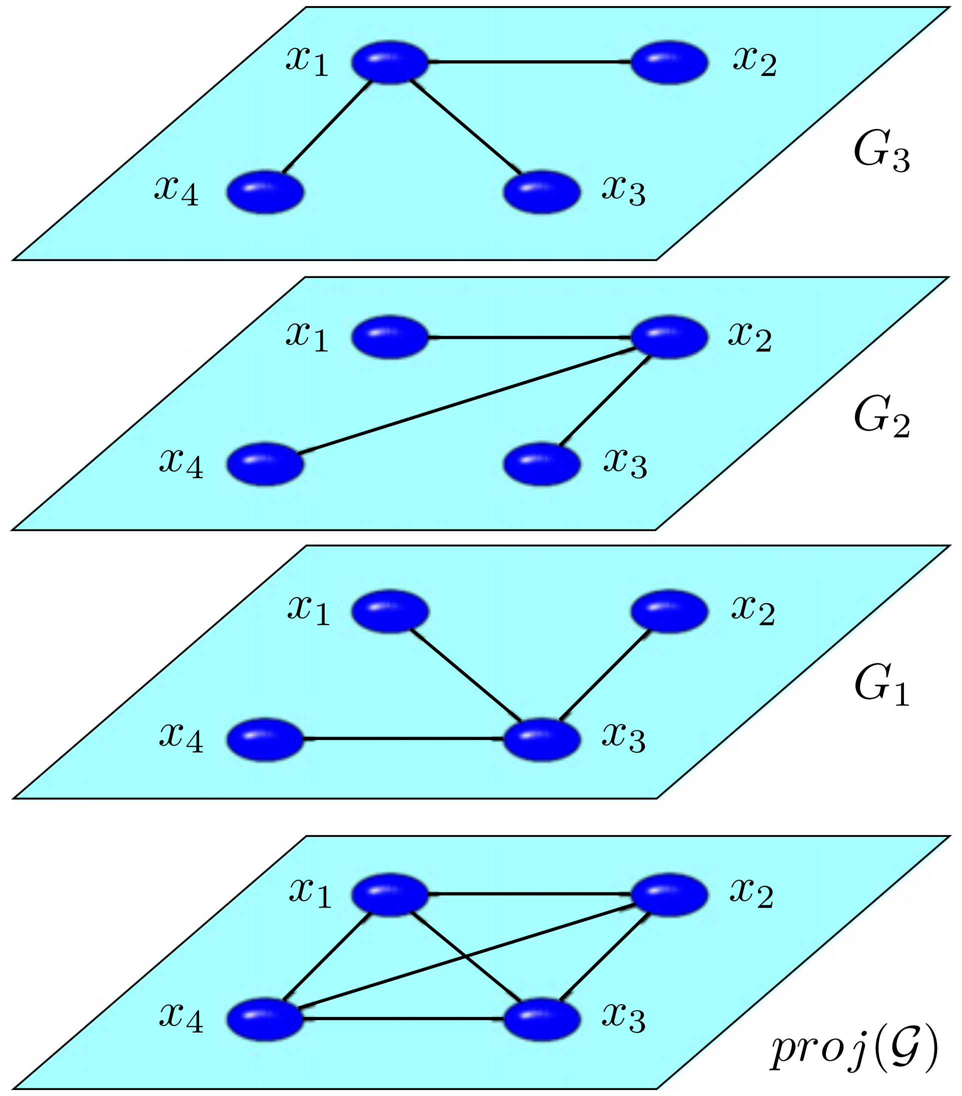
$$A_{\alpha}^{\star} = \sum_{\beta=1}^{M} w_{\alpha\beta} (A^{[\beta]})^{\mathrm{T}}.$$

Clustering coefficient

$$\mathbf{C}_{\mathcal{M}}(i) = \frac{2\sum_{\alpha=1}^{M} |\overline{E_{\alpha}}(i)|}{\sum_{\alpha=1}^{M} |\mathcal{N}_{\alpha}(i)|(|\mathcal{N}_{\alpha}(i)|-1)}.$$

Layer clustering coefficient

$$\mathbf{c}_{\mathcal{M}}^{ly}(i) = \frac{2\sum_{\alpha=1}^{M} |E_{\alpha}(i)|}{\sum_{\alpha=1}^{M} |\mathcal{N}_{\alpha}^{*}(i)|(|\mathcal{N}_{\alpha}^{*}(i)|-1)}.$$



Multilayer Networks

Observables

Walks

$$\{x_1^{\alpha_1}, \ell_1, x_2^{\alpha_2}, \ell_2, \ldots, \ell_{q-1}, x_q^{\alpha_q}\},\qquad \ell_r = \left(x_r^{\alpha_r}, x_{r+1}^{\alpha_{r+1}}\right) \in \mathcal{E}.\qquad \mathcal{E} \in E(\mathcal{M})$$

$$E(\mathcal{M}) = \{E_1, \ldots, E_M\} \bigcup \mathcal{C}.$$

Characteristic path length

$$L(\mathcal{M}) = \frac{1}{N(N-1)} \sum_{\substack{u,v \in X_{\mathcal{M}} \\ u \neq v}} d_{uv},$$

Efficiency

$$E(\mathcal{M}) = \frac{1}{N(N-1)} \sum_{\substack{u,v \in X_{\mathcal{M}} \\ u \neq v}} \frac{1}{d_{uv}}.$$

Interdependence

$$\lambda_i = \sum_{j \neq i} \frac{\psi_{ij}}{\sigma_{ij}},$$

σ_{ij} = # shortest paths between ij

ψ_{ij} = # shortest paths between ij in >2 layers

1 when all shortest paths use edges in at least two layers
0 when all shortest paths use only one layer of the system.

Supra-laplacian for multilayer networks

$$\mathcal{L} = \begin{pmatrix} D_1\mathbf{L^1} & 0 & \ldots & 0 \\ 0 & D_2\mathbf{L^2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D_M\mathbf{L^M} \end{pmatrix} + \begin{pmatrix} \sum_\beta D_{1\beta}\mathbf{I} & -D_{12}\mathbf{I} & \ldots & -D_{1M}\mathbf{I} \\ -D_{21}\mathbf{I} & \sum_\beta D_{2\beta}\mathbf{I} & \ldots & -D_{2M}\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ -D_{M1}\mathbf{I} & -D_{M2}\mathbf{I} & \ldots & \sum_\beta D_{M\beta}\mathbf{I} \end{pmatrix}.$$

Multilayer Networks

Correlations

Full characterisation of matrix $P(k^\alpha, k^\beta)$

$$P(k^\alpha, k^\beta) = \frac{N(k^\alpha, k^\beta)}{N},$$

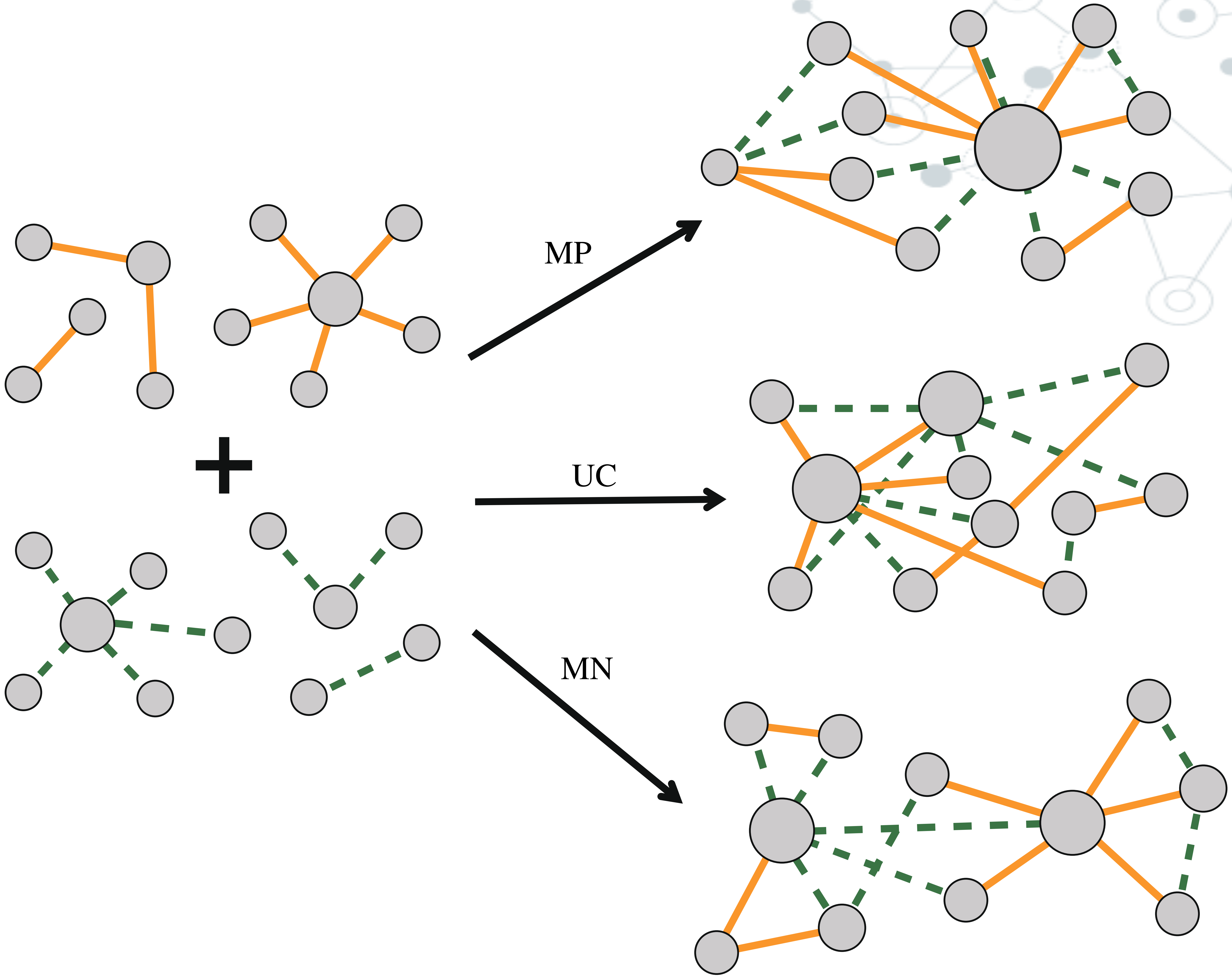
Average degree in layer α conditioned on the degree of the node in layer β

$$\bar{k}^\alpha(k^\beta) = \sum_{k^\alpha} k^\alpha P(k^\alpha | k^\beta) = \frac{\sum_{k^\alpha} k^\alpha P(k^\alpha, k^\beta)}{\sum_{k^\alpha} P(k^\alpha, k^\beta)}.$$

Spearman degree correlations

$$r_{\alpha\beta} = \frac{\left\langle k_i^{[\alpha]} k_i^{[\beta]} \right\rangle - \left\langle k_i^{[\alpha]} \right\rangle \left\langle k_i^{[\beta]} \right\rangle,}{\sigma_\alpha \sigma_\beta}$$

$$\sigma_\alpha = \sqrt{\left\langle k_i^{[\alpha]} k_i^{[\alpha]} \right\rangle - \left\langle k_i^{[\alpha]} \right\rangle^2}.$$



Multilayer Networks

Multiplexity

Activity of node i in layer alpha: 1 if k[α] i > 0 and 0 otherwise

$$b_{i,\alpha} = 1 - \delta_{0,k_i^{[\alpha]}} = 1 - \delta_{0,\sum_j^N a_{ij}^{\alpha}},$$

Node activity

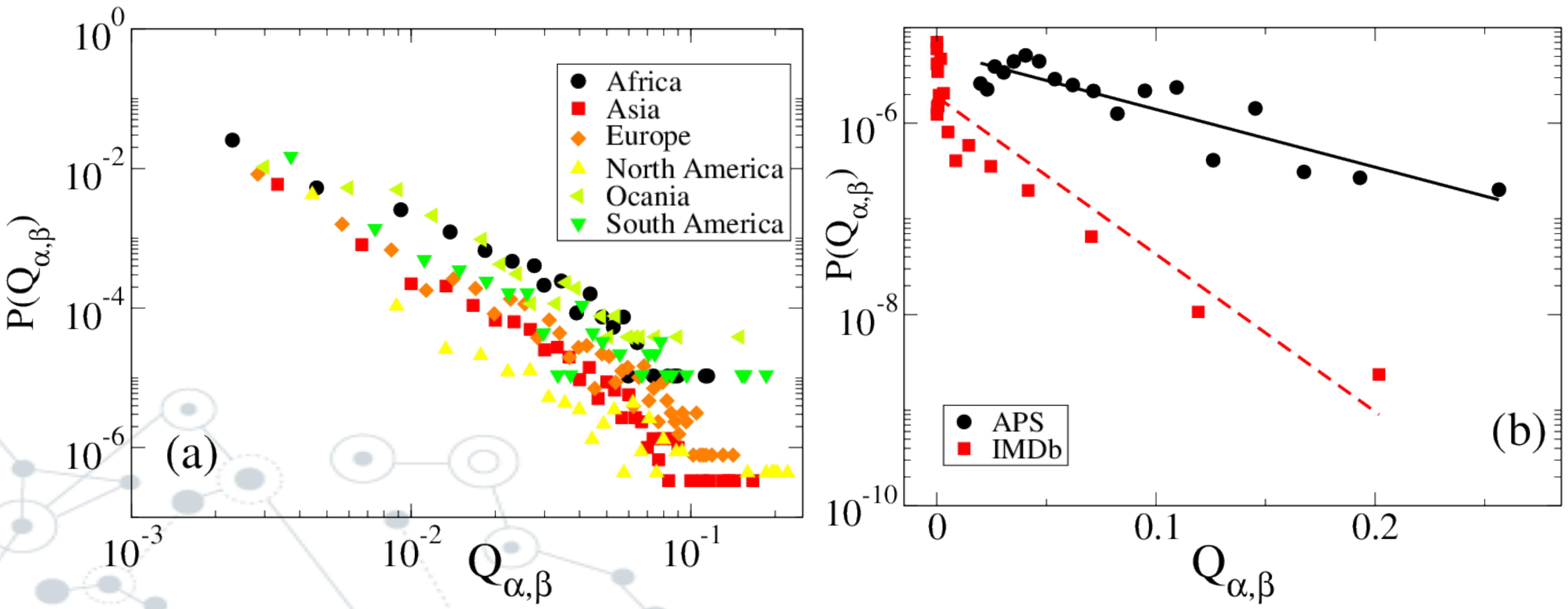
$$B_i = \sum_{\alpha=1}^{M} b_{i,\alpha}.$$

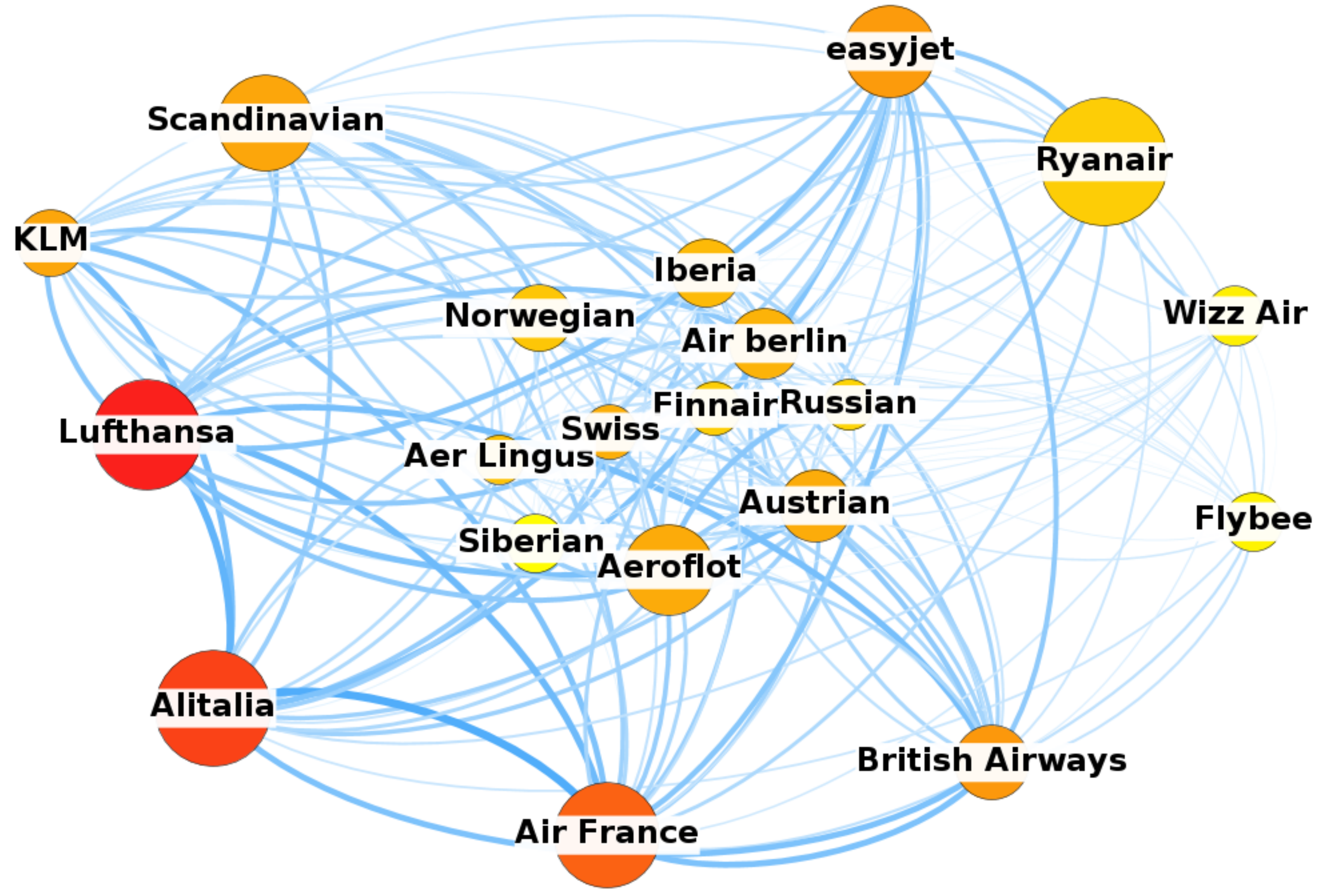
Layer activity

$$N_\alpha = \sum_{i=1}^{N} b_{i,\alpha}.$$

Layer pairwise multiplexity: measuring the correlation between the layers

$$Q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} b_{i,\alpha} b_{i,\beta},$$





Measuring and modeling correlations in multiplex networks

Vincenzo Nicosia[1,*] and Vito Latora[1]

[1]School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom

Multilayer Networks

Reducibility

Von Neumann entropy "Mixedness" (=0 if pure state)

$$h_A = -\mathrm{Tr}\left[\mathcal{L}_G \log_2 \mathcal{L}_G\right] \qquad h_A = -\sum_{i=1}^{N} \lambda_i \log_2(\lambda_i),$$

$$\mathcal{L}_G = c \times (D - A) \qquad \mathrm{Tr}(\mathcal{L}_G) = 1$$

$$c = 1/(\textstyle\sum_{i,j \in V} a_{ij}) = \frac{1}{2K}$$

1 layer - 1 "state"

Reduction

$$\mathcal{A} = \{A_1, A_2, \ldots, A_M\}$$

Aggregate some of the layers

$$\mathcal{C} = \{C_1, C_2, \ldots, C_X\} \quad X < M$$

VN entropy of multilayer network

$$\bar{H}(\mathcal{C}) = \frac{H(\mathcal{C})}{X} = \frac{\sum_{\alpha=1}^{X} h_{C^{[\alpha]}}}{X}$$

Relative entropy

Larger if more distinguishable from fully aggregated

$$q(\mathcal{C}) = 1 - \frac{\bar{H}(\mathcal{C})}{h_A}$$

Entropy Aggregated graph

Mopt corresponds to argmax q(C)

$$\chi(\mathcal{A}) = \frac{M - M_{\mathrm{opt}}}{M - 1},$$

Reducibility

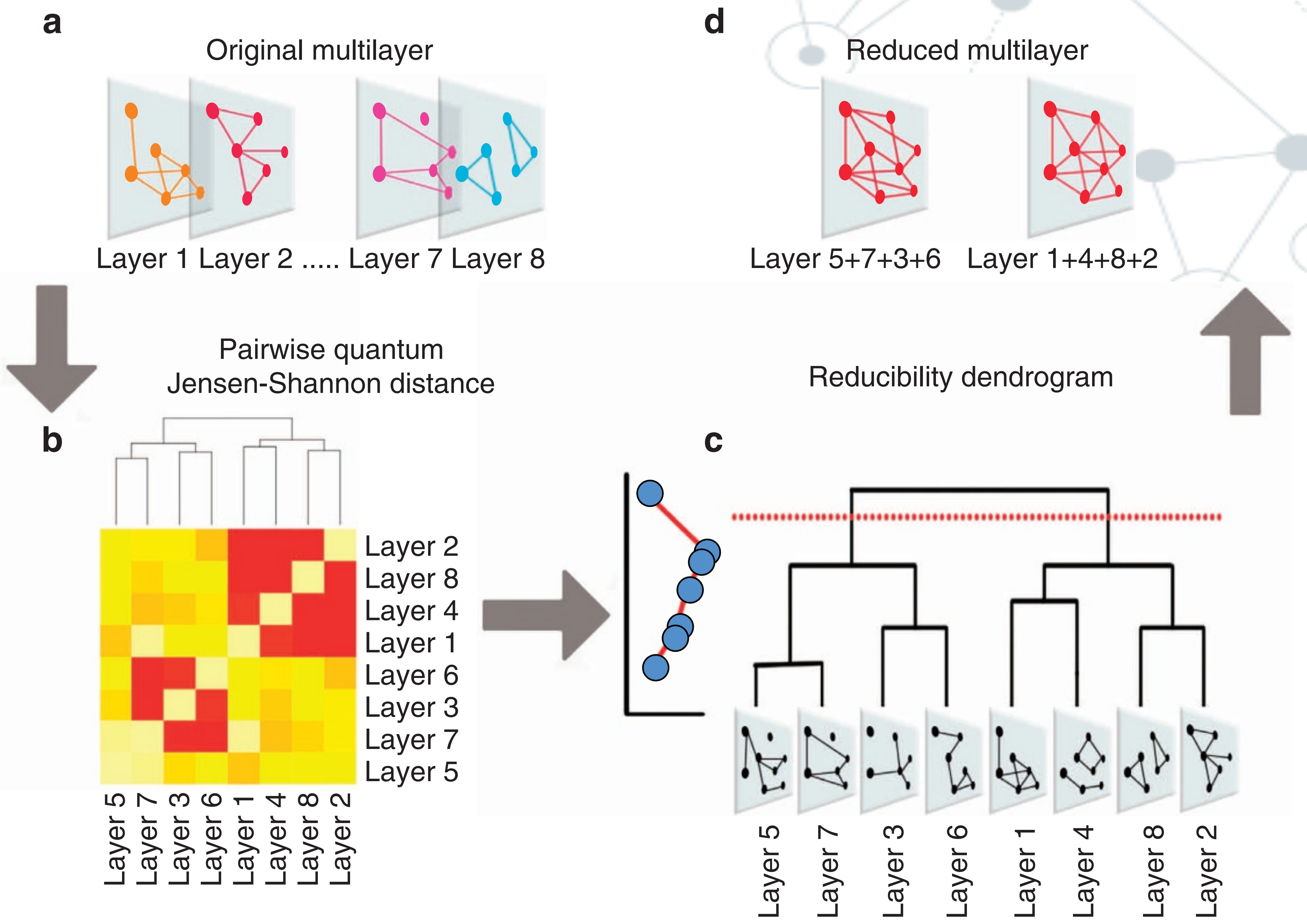
0 if cannot be reduced
1 if reducible to single layer

ARTICLE

Received 3 Jul 2014 | Accepted 7 Mar 2015 | Published 23 Apr 2015 DOI: 10.1038/ncomms7864

Structural reducibility of multilayer networks

Manlio De Domenico[1,*], Vincenzo Nicosia[2,*], Alexandre Arenas[1] & Vito Latora[2,3]



Kullback-Leibler divergence

$$\mathcal{D}_{KL}(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \mathrm{Tr}\left[\boldsymbol{\rho}\left(\log_2(\boldsymbol{\rho}) - \log_2(\boldsymbol{\sigma})\right)\right]$$

Jensen-Shannon divergence

$$\mathcal{D}_{\mathrm{JS}}(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \frac{1}{2}\mathcal{D}_{\mathrm{KL}}(\boldsymbol{\rho}||\boldsymbol{\mu}) + \frac{1}{2}\mathcal{D}_{\mathrm{KL}}(\boldsymbol{\sigma}||\boldsymbol{\mu}) = h(\boldsymbol{\mu}) - \frac{1}{2}\left[h(\boldsymbol{\rho}) + h(\boldsymbol{\sigma})\right].$$

$$\boldsymbol{\mu} = \frac{1}{2}(\boldsymbol{\rho} + \boldsymbol{\sigma})$$

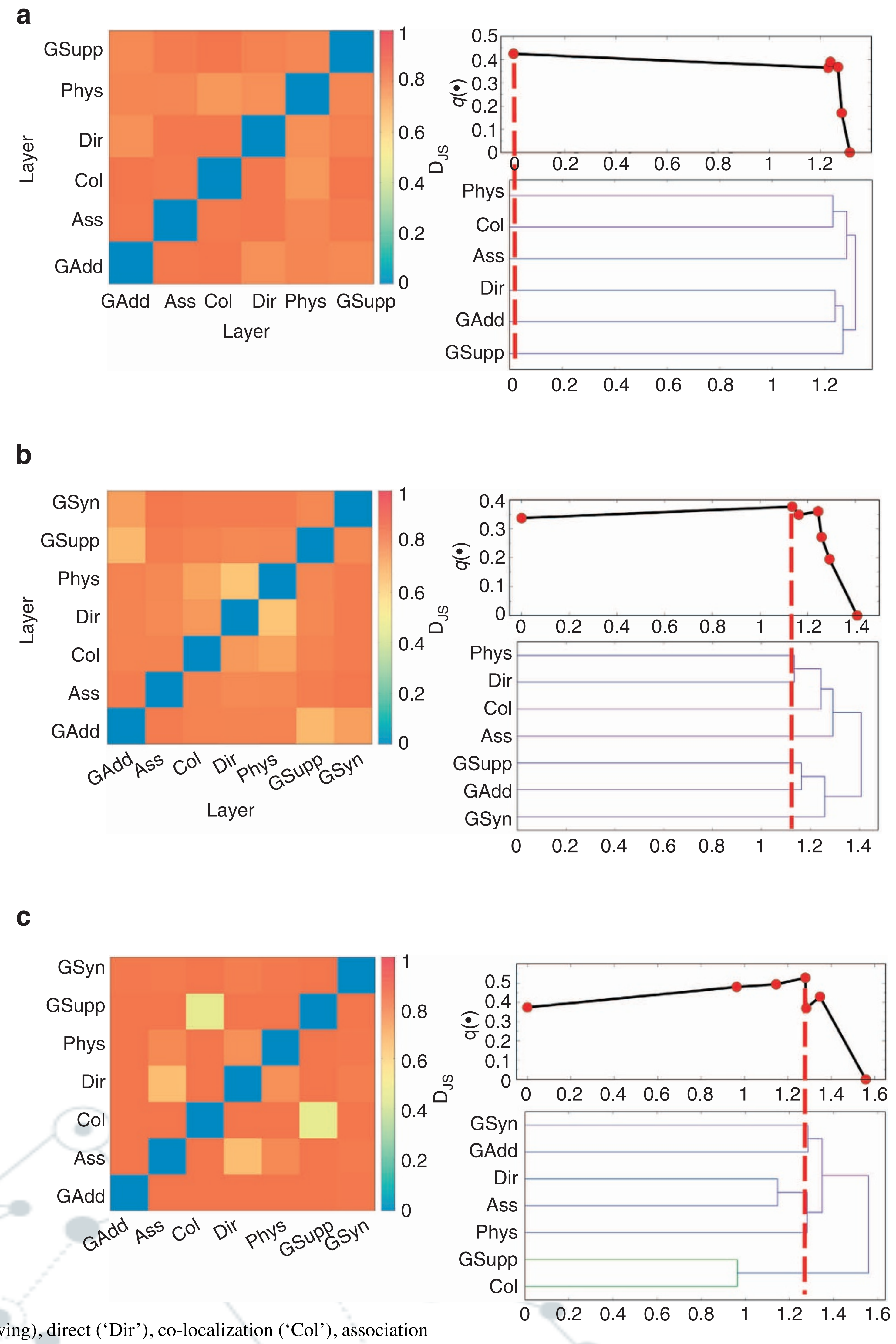
Multilayer Networks

Reducibility

C. Elegans

Mus

Yeast

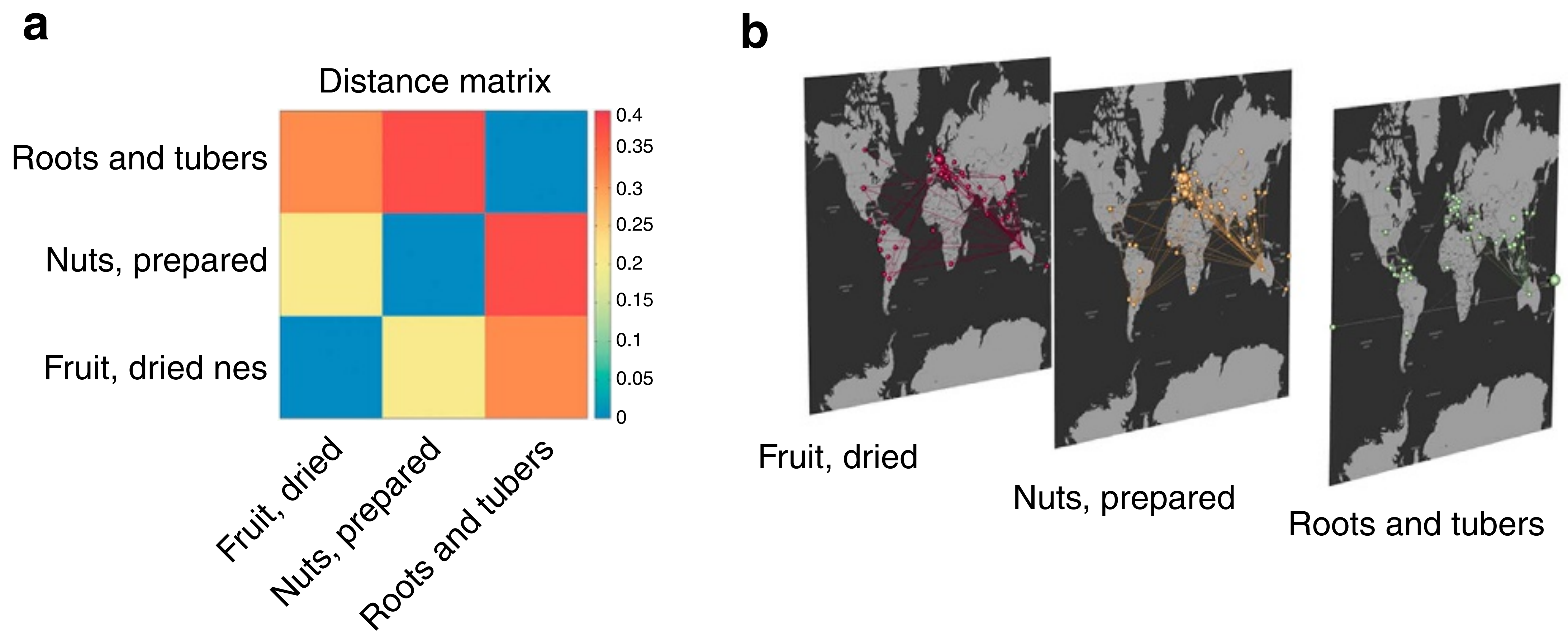


physical (labelled ‘Phys’ in the following), direct (‘Dir’), co-localization (‘Col’), association (‘Ass’) and suppressive (‘GSup’), additive (‘GAdd’) or synthetic genetic (‘GSyn’) interaction.

Table 1 | Reducibility of empirical multilayer networks.

Network	N	M	M_{opt}	max[q(•)]	χ
Arabidopsis	6981	7	5	0.436	0.33
Bos	326	4	3	0.494	0.33
Candida	368	7	4	0.527	0.50
C. elegans	3880	6	4	0.390	0.40
Drosophila	8216	7	5	0.426	0.33
Gallus	314	6	4	0.505	0.40
Human HIV-1	1006	5	2	0.499	0.75
Mus	7748	7	6	0.376	0.17
Plasmodium	1204	3	2	0.500	0.50
Rattus	2641	6	4	0.504	0.40
S. cerevisiae	6571	7	4	0.115	0.50
S. pombe	4093	7	4	0.197	0.50
Xenopus	462	5	3	0.424	0.50
Arxiv coauthorship	14065	13	11	0.231	0.17
Terrorist network	78	4	2	0.239	0.67
FAO Trade network	184	340	182	0.354	0.47
London Tube	369	13	12	0.441	0.08
Airports Europe	1064	175	165	0.667	0.06
Airports Asia	1130	213	202	0.653	0.05
Airports North America	2040	143	136	0.686	0.05

Number of nodes (N), number of layers in the original system (M), number of layers (M_{opt}) corresponding to the maximal value of the quality function (max[q(•)]) obtained through the greedy hierarchical clustering procedure, and the value of the reducibility (χ) for several biological, social, economical and technological multilayer networks. Notice that the structure of the three continental air networks and of the London metropolitan transportation system cannot be substantially reduced, in accordance with the fact that in these systems layer redundancy is purposedly avoided. Conversely, social and biological systems exhibit higher levels of redundancy and allow for the merging of up to 75% of the layers.



Multilayer Networks

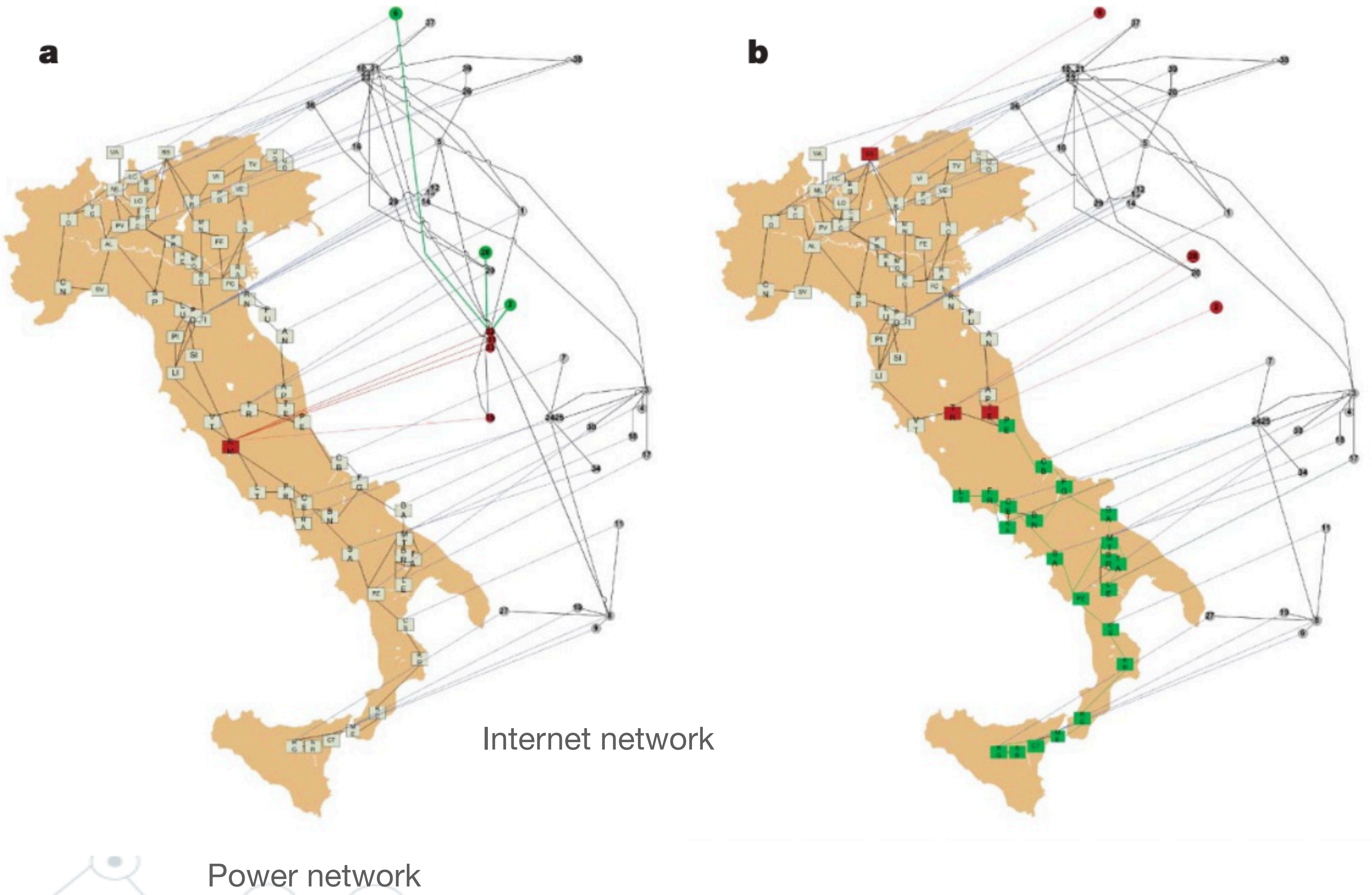
Resilience: Modelling a blackout in Italy (September 2003)

Letter | Published: 15 April 2010

Catastrophic cascade of failures in interdependent networks

Sergey V. Buldyrev, Roni Parshani, Gerald Paul, H. Eugene Stanley & Shlomo Havlin

Nature **464**, 1025–1028 (2010) | Cite this article



Multilayer Networks

Resilience

Nodes are layers can be interdependent: failure in one induces failure in the other

Letter | Published: 15 April 2010

Catastrophic cascade of failures in interdependent networks

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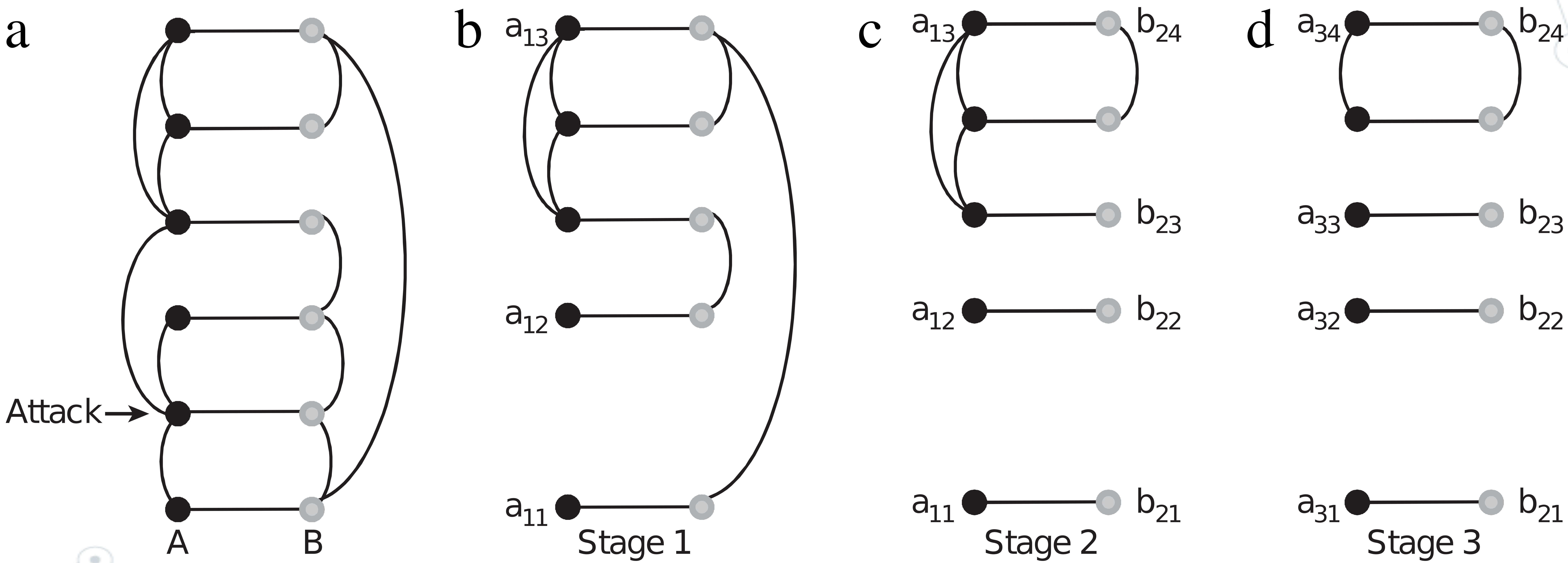

find the mutually connected components

in presence of interdependencies, the robustness of multilayer networks can be evaluated by calculating the size of their mutually connected giant component (MCGC)

New result:
Multilayer SF are less resilient!

Code

https://github.com/nkoub/multinetx
https://github.com/bolozna/Multilayer-networks-library
https://github.com/manlius/muxViz

Networks with higher-order (group) interactions

Hypergraphs and simplicial complexes

Physics Reports 874 (2020) 1–92

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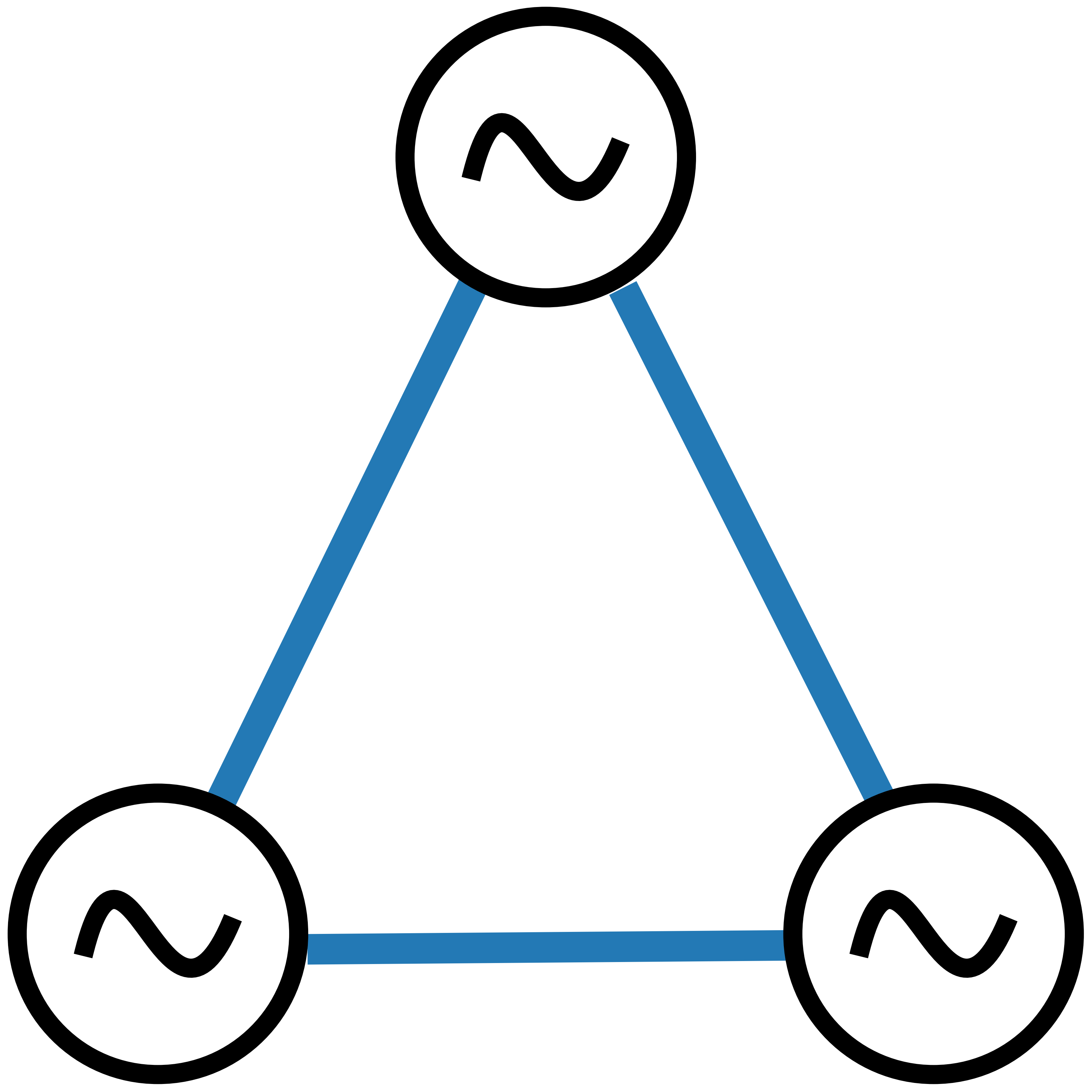
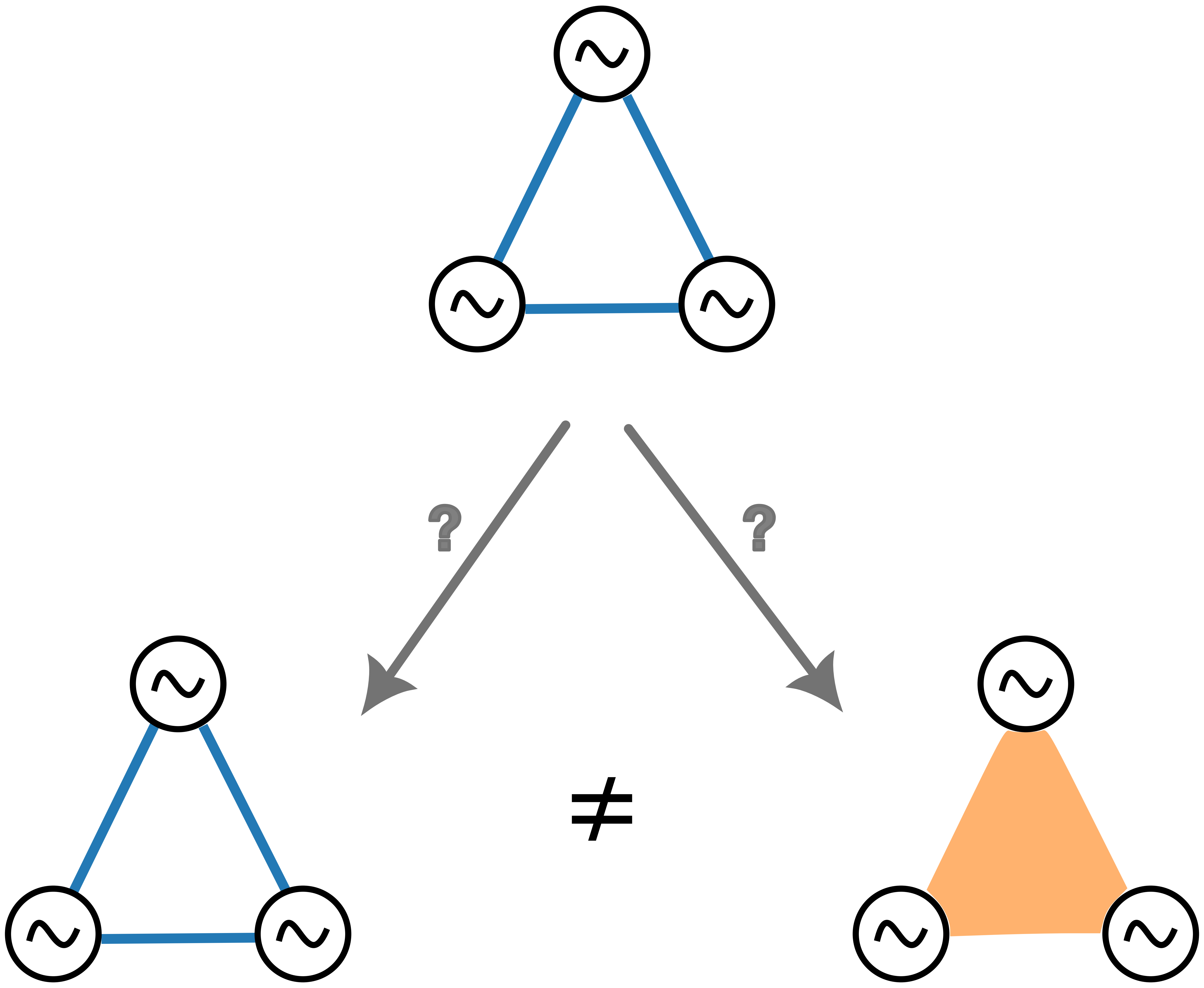
Networks beyond pairwise interactions: Structure and dynamics

Federico Battiston [a,*], Giulia Cencetti [b], Iacopo Iacopini [c,d], Vito Latora [c,e,f,g], Maxime Lucas [h,i,j], Alice Patania [k], Jean-Gabriel Young [l], Giovanni Petri [m,n]

(Pairwise) networks are great



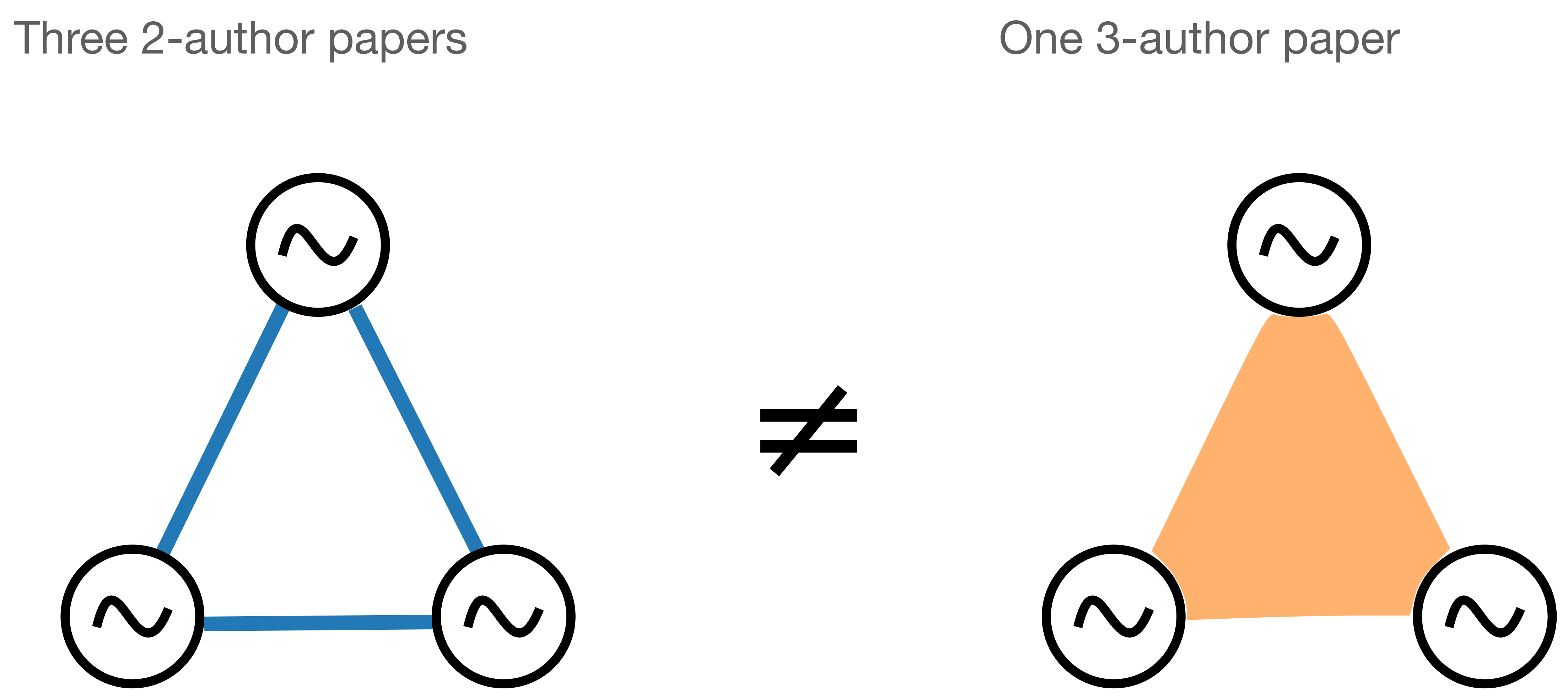
But they don't encode group interactions



Going beyond pairwise

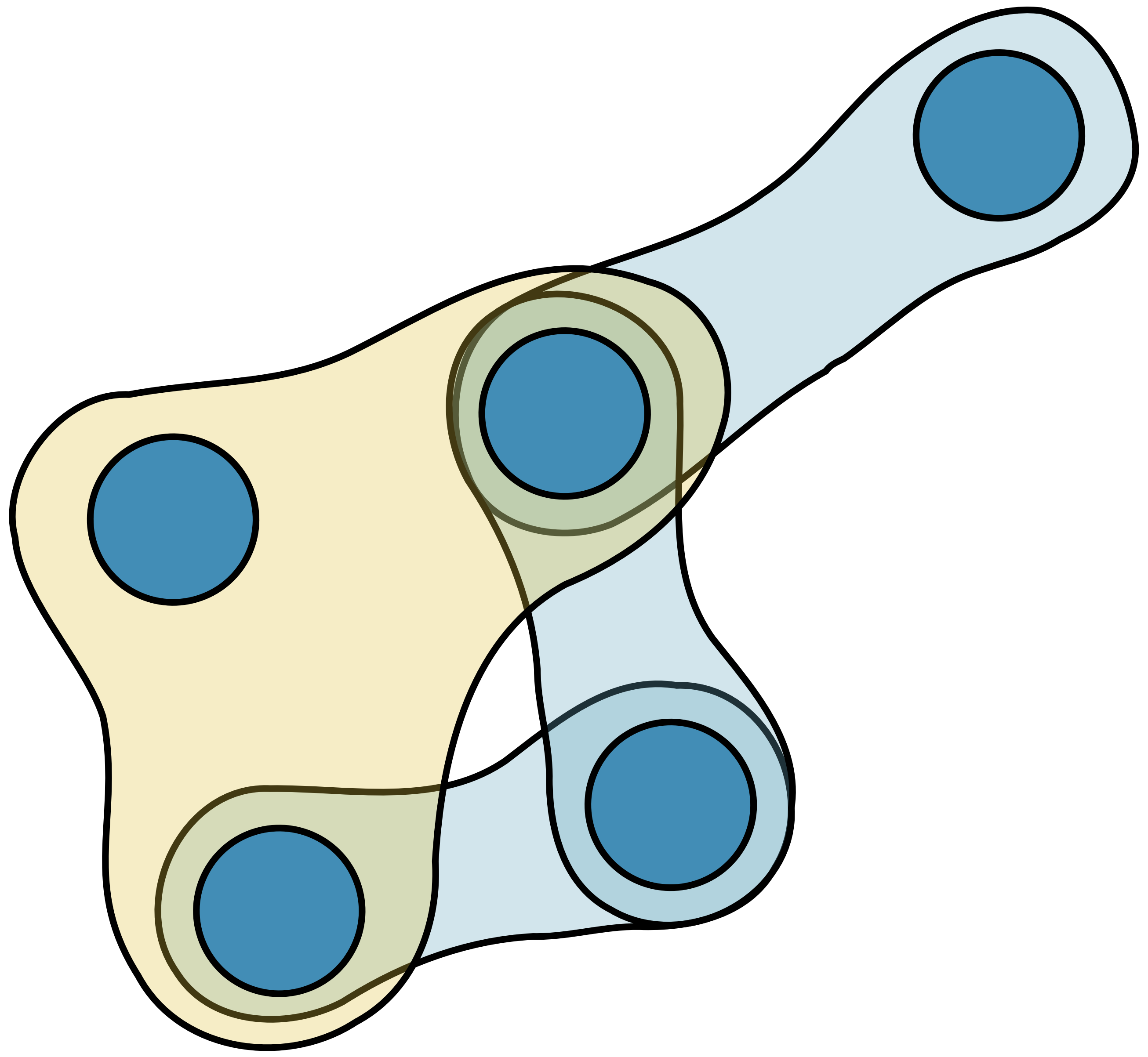
Examples

- Co-authorship
- Chemical reactions
- Social interactions
- Etc.



Two possible representations

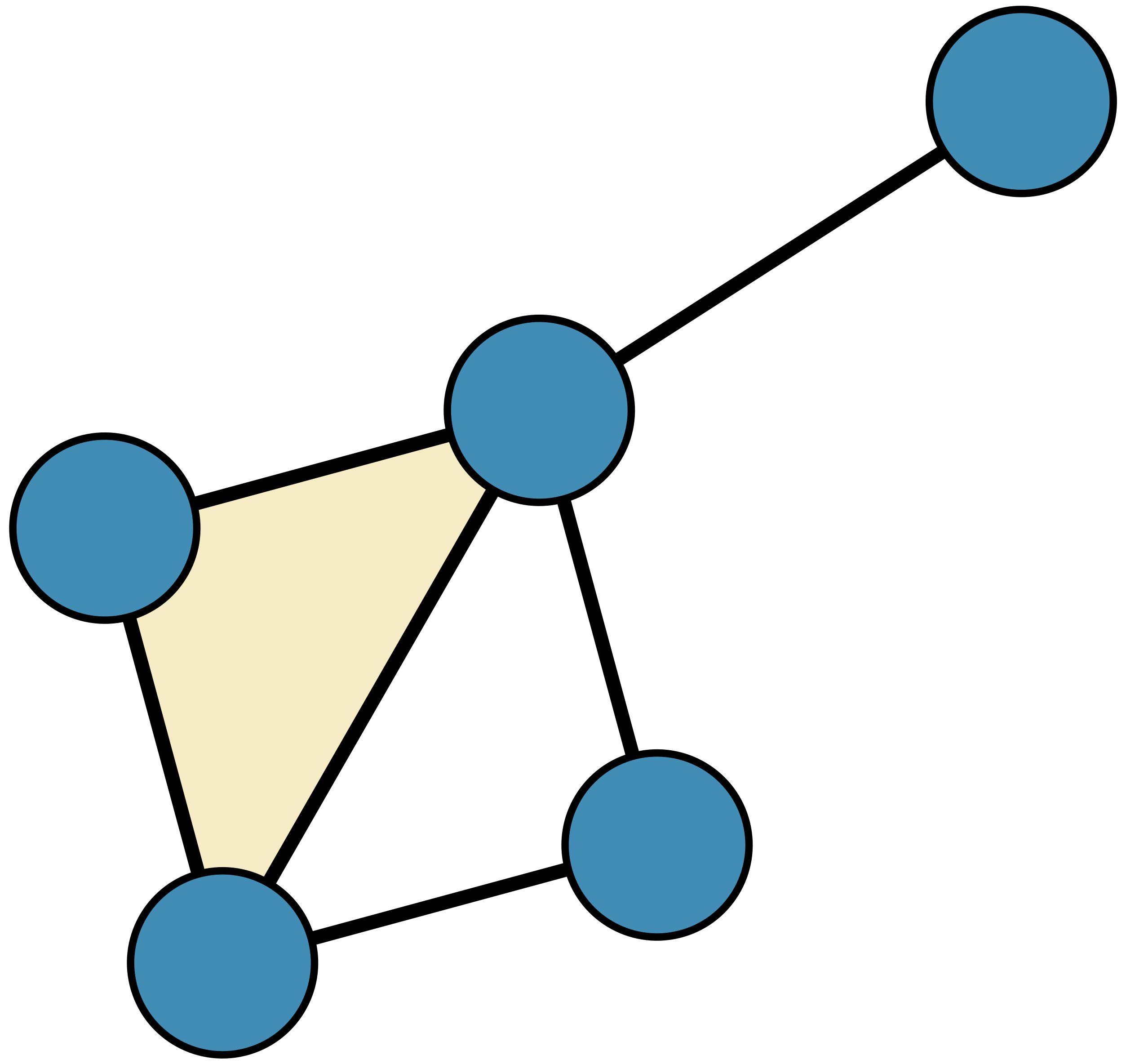
Hypergraphs



Definition: (V, E) set of nodes V and hyper edges E

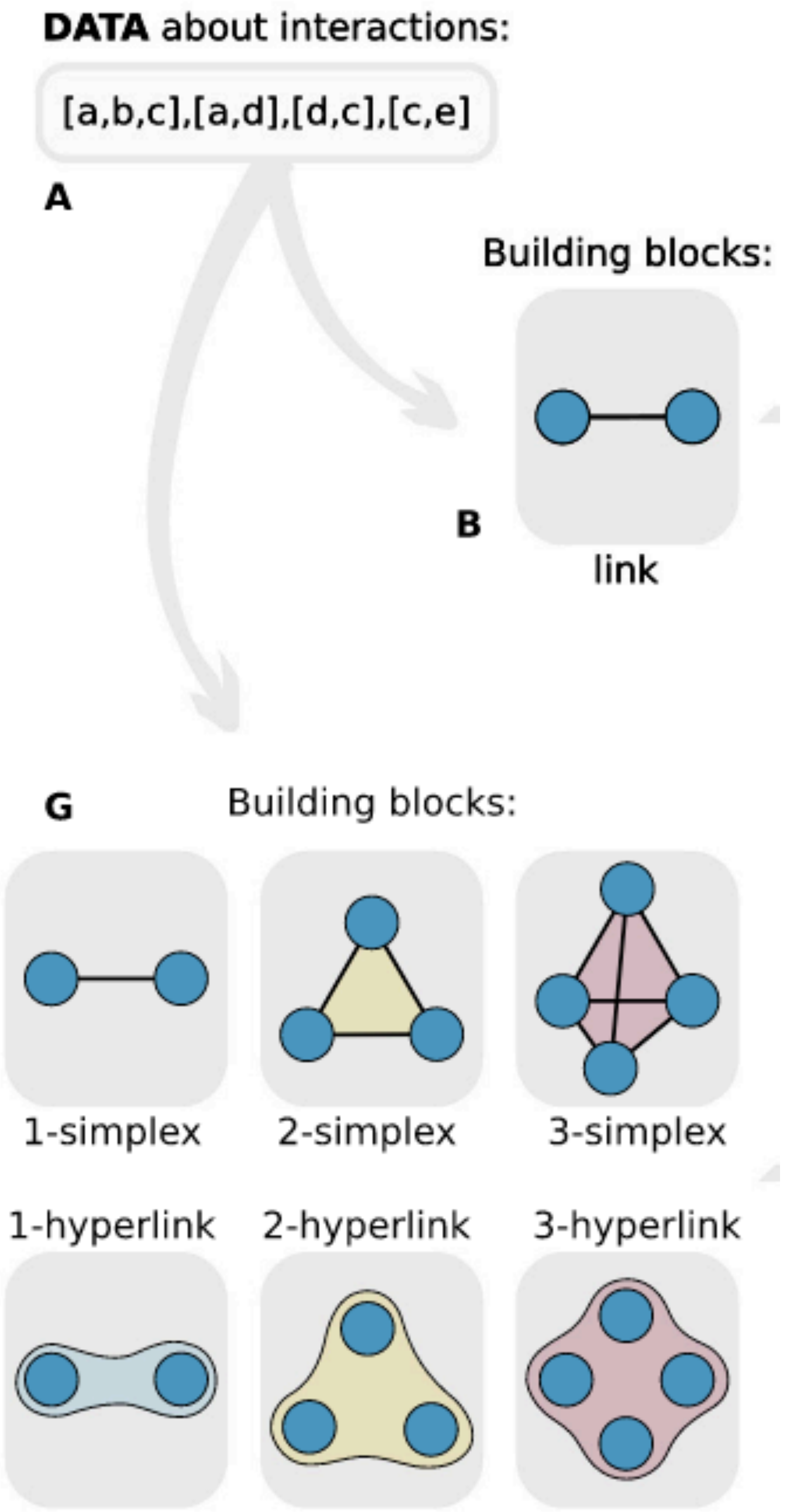
A hyper edge is a set of any number of nodes e.g. {1, 2, 3}

Simplicial complexes



Special case of hyper graphs with one extra condition:
All subfaces must be included

Building blocks



Network

Simplicial complex

Hypergraph

(Hyperlink = hyper edge)

“Order” of interaction = size - 1

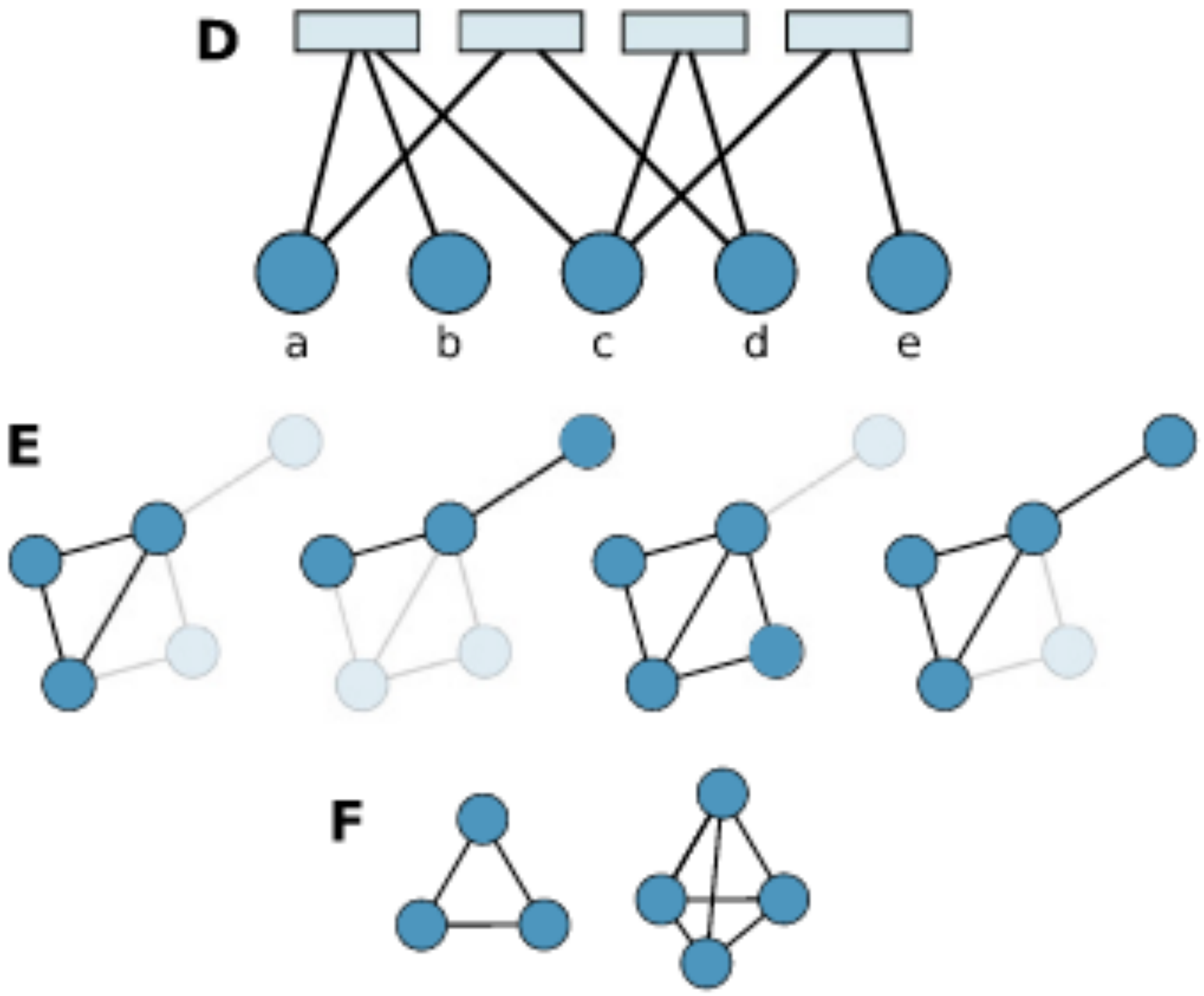
Link with other types networks

Bipartite, motifs, and multilayers

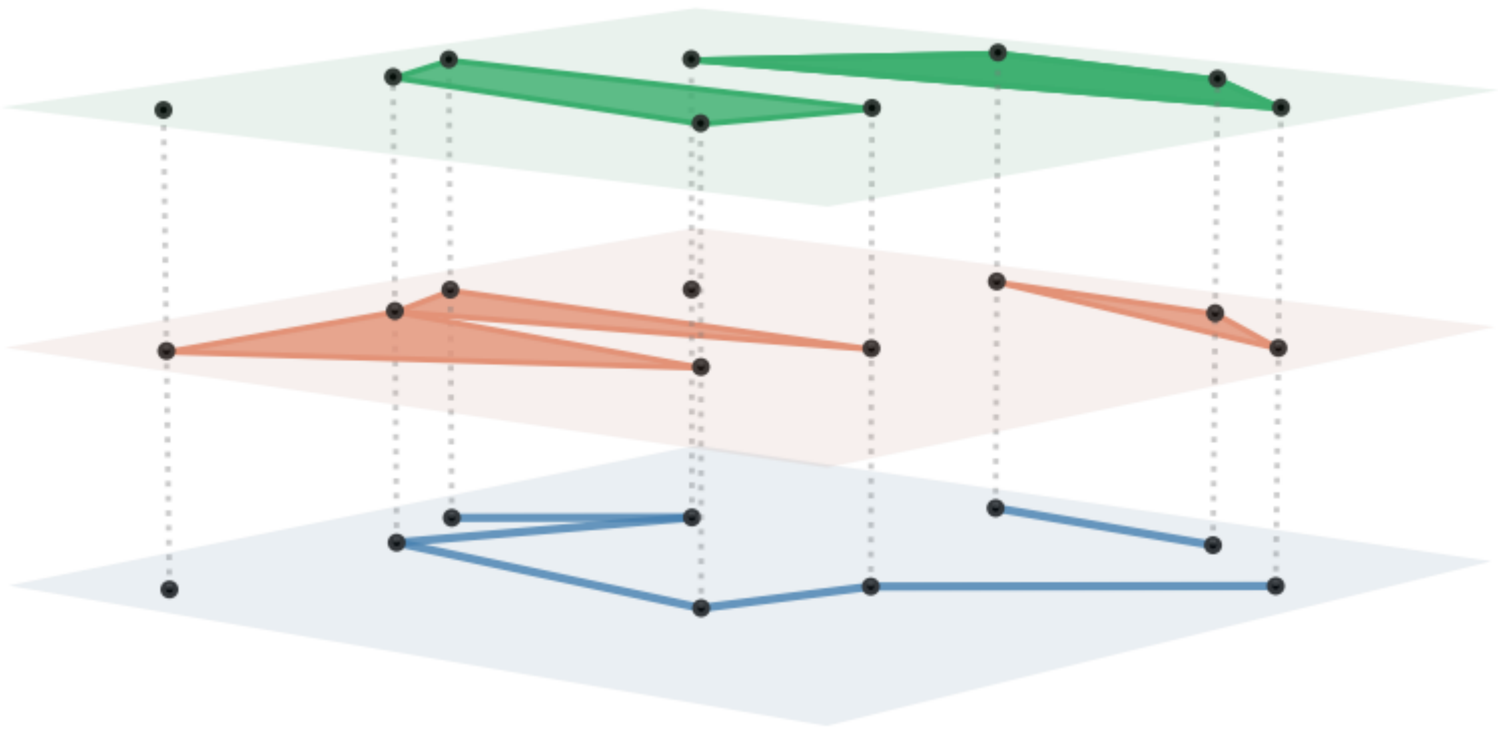
> BIPARTITE GRAPH
The top layer describes groups

> NETWORK MOTIFS

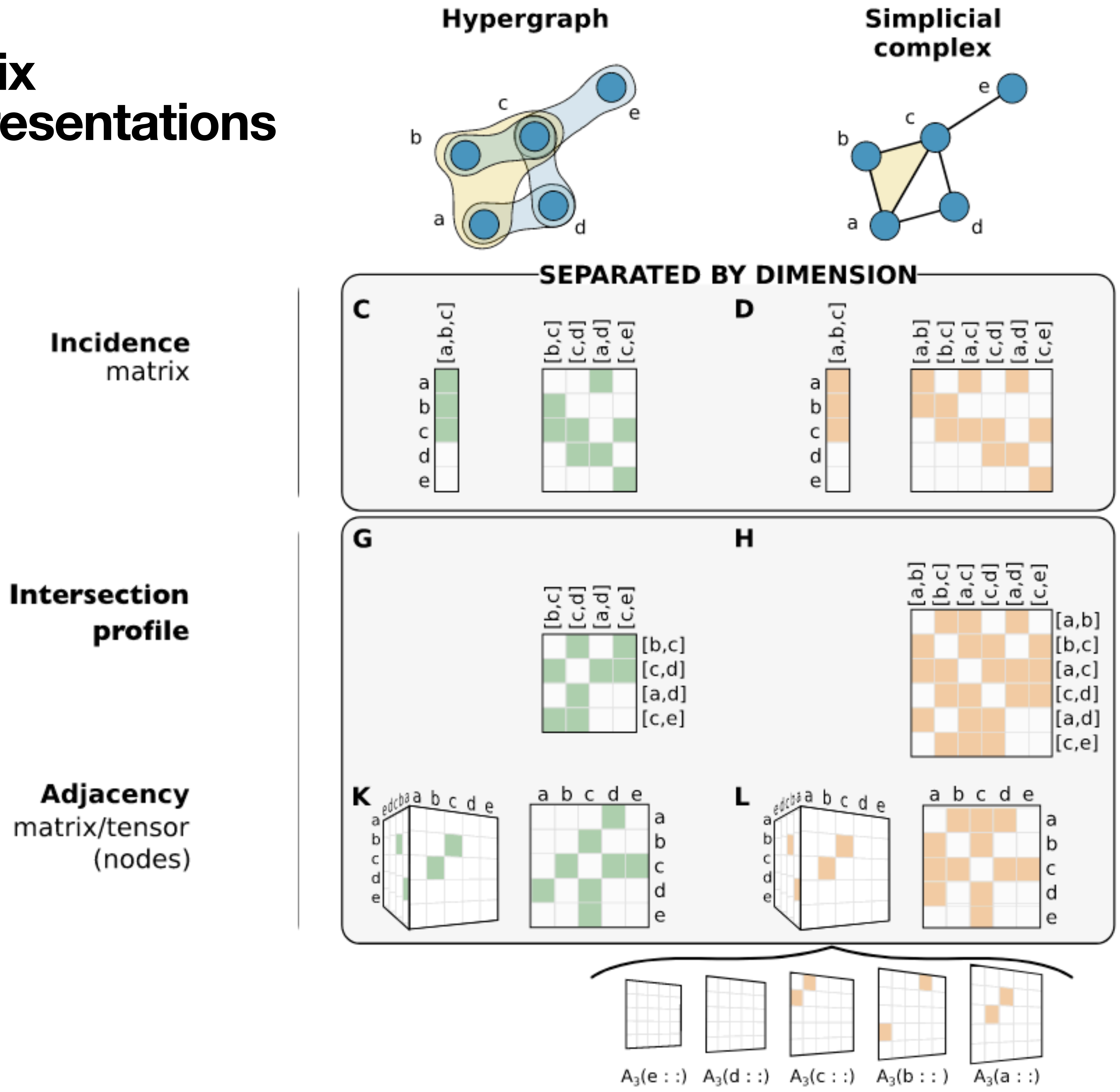
> CLIQUES
Special type of motifs



“Multilayer”



Matrix Representations



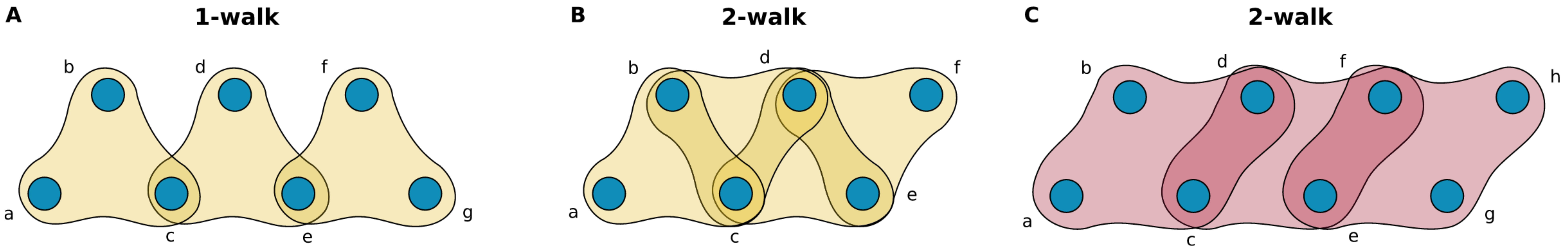
I_iα in row i and column α is 1 if node i and edge α are incident, and zero otherwise

$$P = I^T I,$$

$$A = II^T - D$$

Measures

- Degree
- Walks



Current research

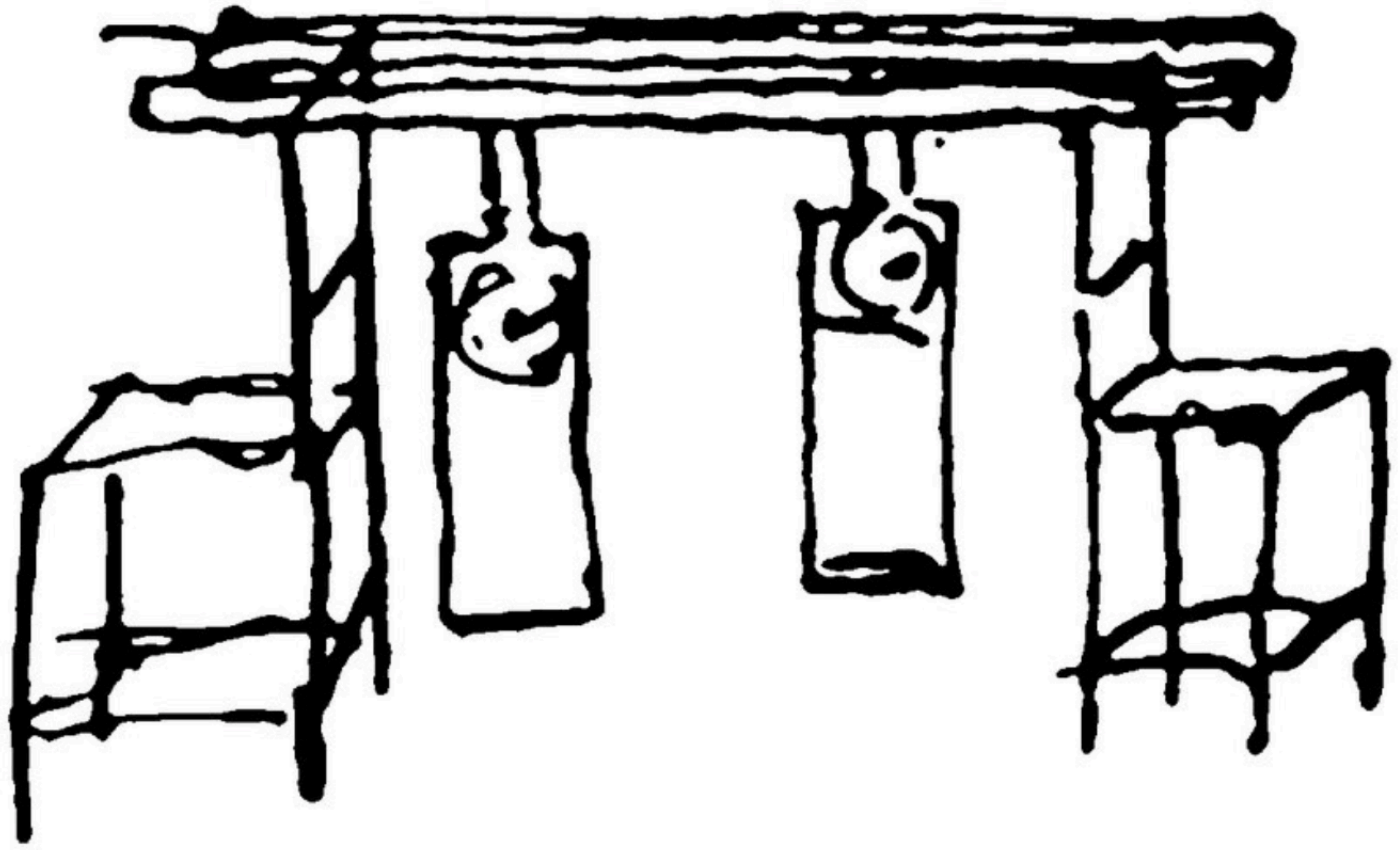
- Models and phenomenology (sync, contagion, etc)
- Reducibility?
- Information theory: new scales?
- Coupling functions
- XGI
- …

Before showing you: some synchronization

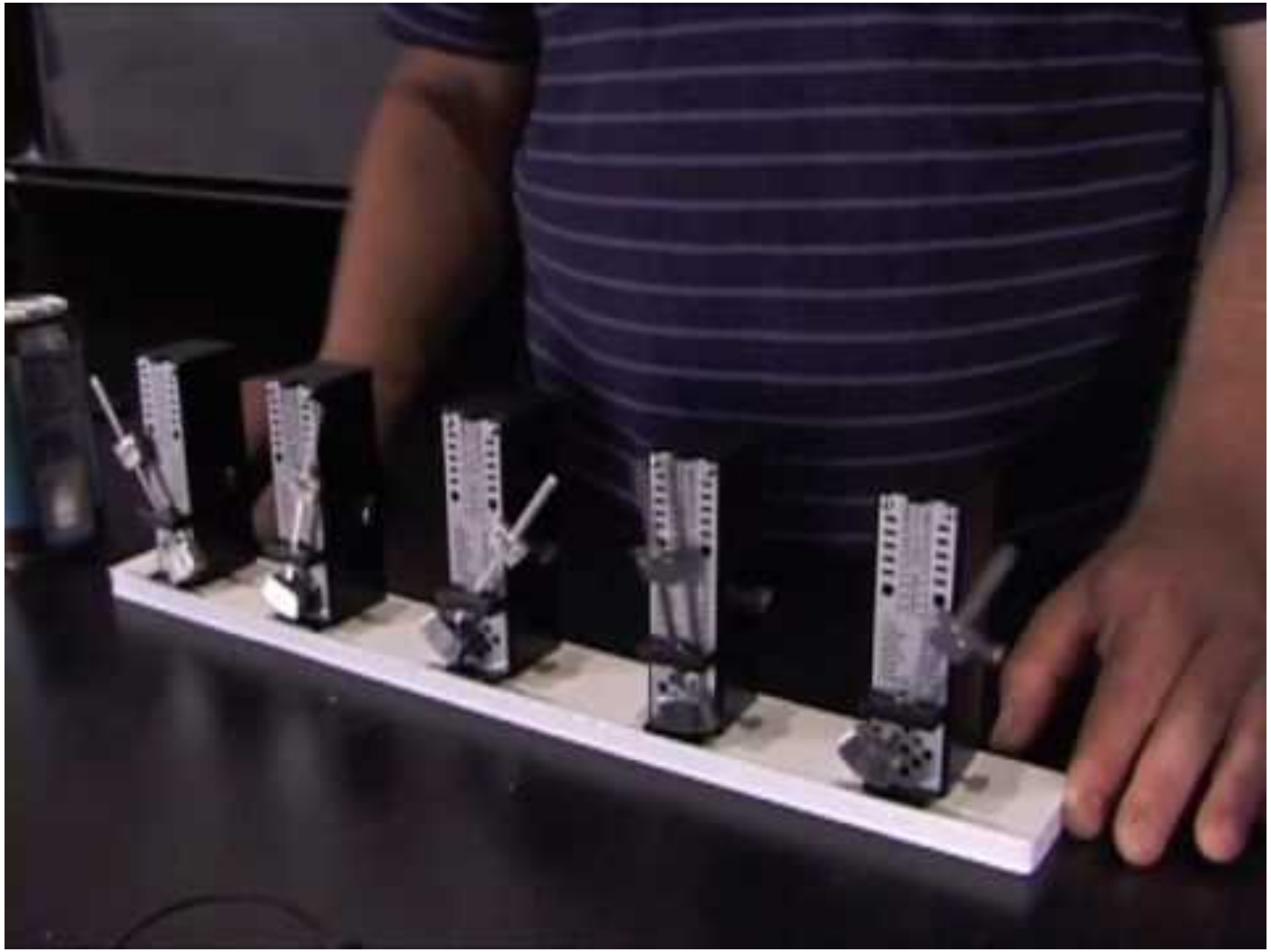
Synchronization

Story time: Christiaan Huygens (XVII)

noticed that two mechanical clocks when attached to a beam synchronize the movement of their pendula.



Experiment with metronomes



What is needed for sync?

Sync everywhere in nature

Metronomes can by any oscillator or rhythms

Examples:

- neurons firing
- Circadian rhythms
- fireflies flashing

Refs:
"Sync: The Emerging Science of Spontaneous Order" by Steven Strogatz
"Synchronization: A Universal Concept in Nonlinear Sciences" by Pikovsky, Rosenblum, and Kurths

Simplest oscillator: just a phase

$$\dot{\theta} = \omega$$

has a constant frequency.

Best visualized in the x-y-plane as a dot the moves around in a circle at constant speed.

Minimal case for sync: 2 oscillators

$$\dot{\theta}_1 = \omega_1 + \frac{\gamma}{2}\sin(\theta_2 - \theta_1)$$

$$\dot{\theta}_2 = \omega_2 + \frac{\gamma}{2}\sin(\theta_1 - \theta_2)$$

Natural frequencies w1 and w2
Coupling strength \gamma

Condition for sync: constant phase diff

We define the phase difference

$$\psi = \theta_2 - \theta_1$$

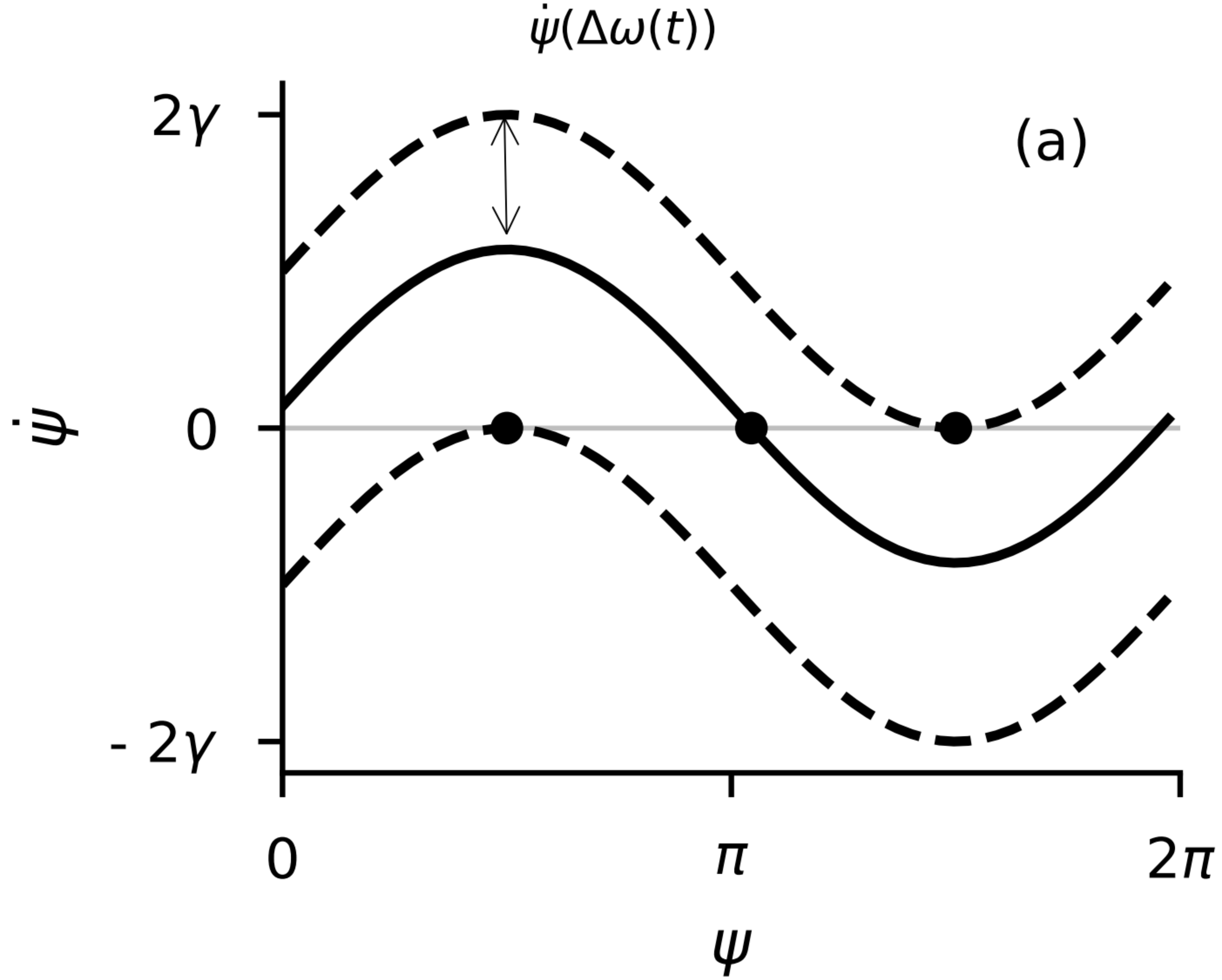
Which evolves as

$$\dot{\psi} = \Delta\omega - \gamma\sin(\psi)$$

With the frequency mismatch $\Delta\omega = \omega_2 - \omega_1$

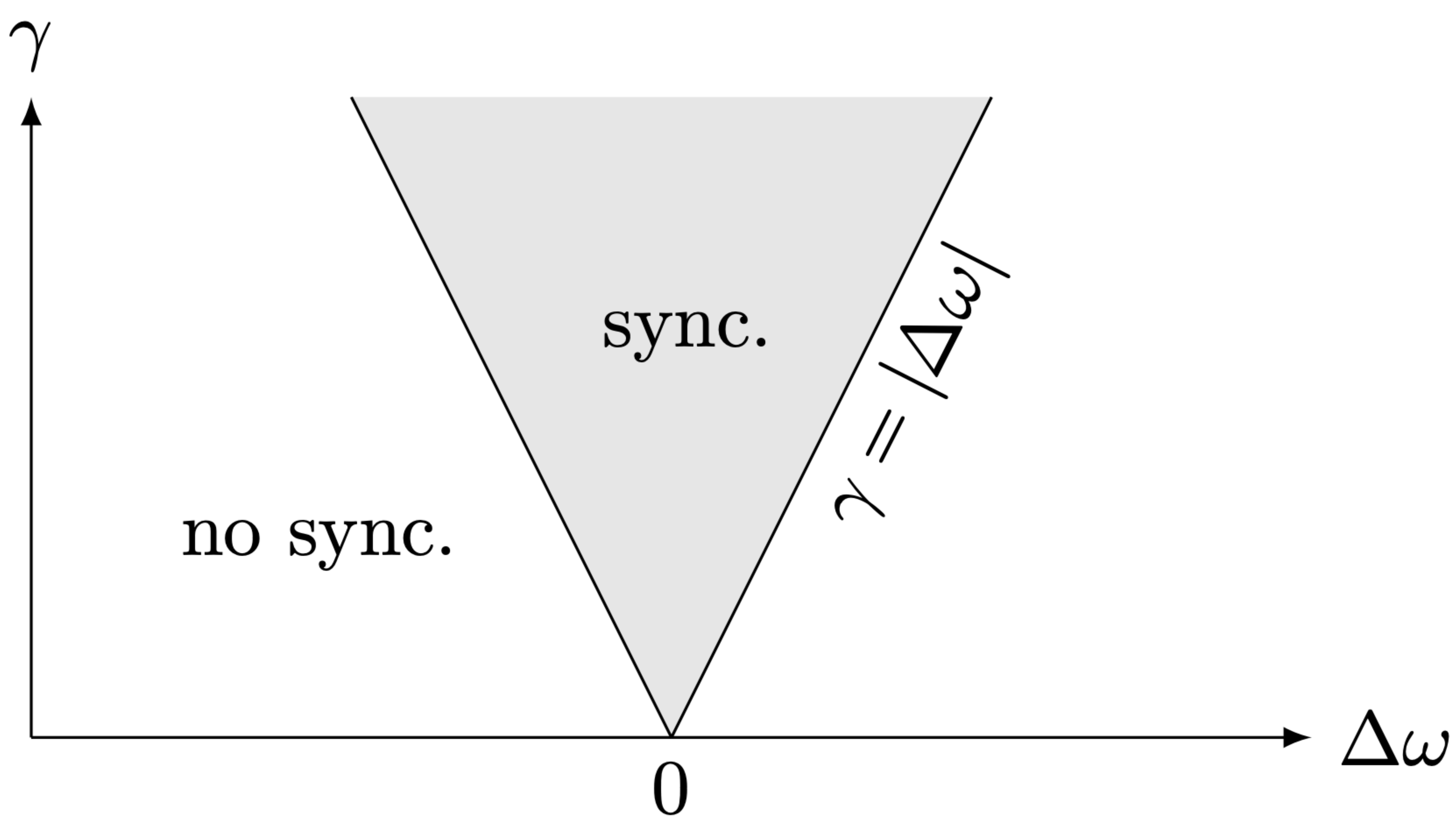
Condition for sync: fixed points

$$\dot{\psi} = \Delta\omega - \gamma \sin(\psi) \equiv 0$$



Condition for sync:
large coupling strength or small frequency mismatch

$$\Rightarrow \gamma > |\Delta\omega|$$

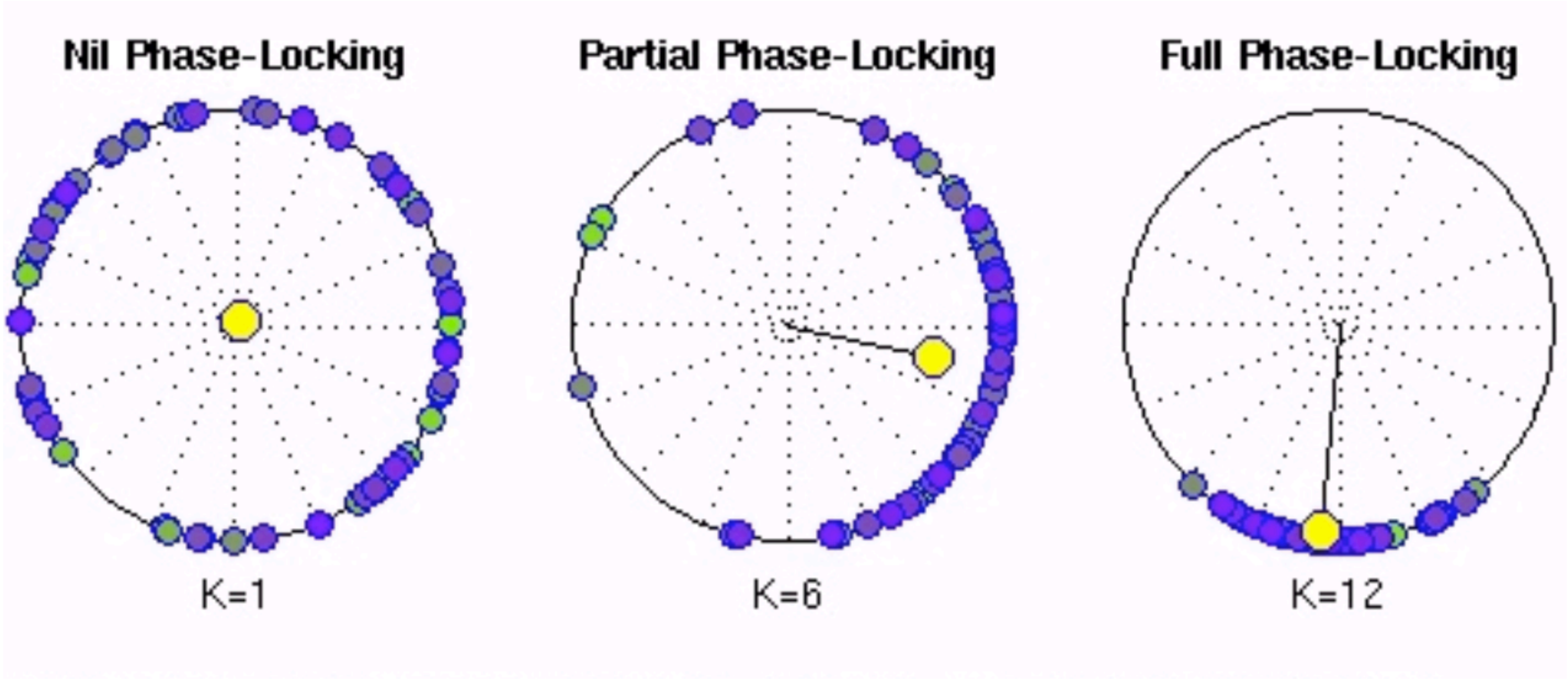


Many oscillators: Kuramoto model

$$\dot{\theta}_i = \omega_i + \frac{\gamma}{N} \sum_j A_{ij} \sin(\theta_j - \theta_i)$$

With the adjacency matrix of the network A_{ij}

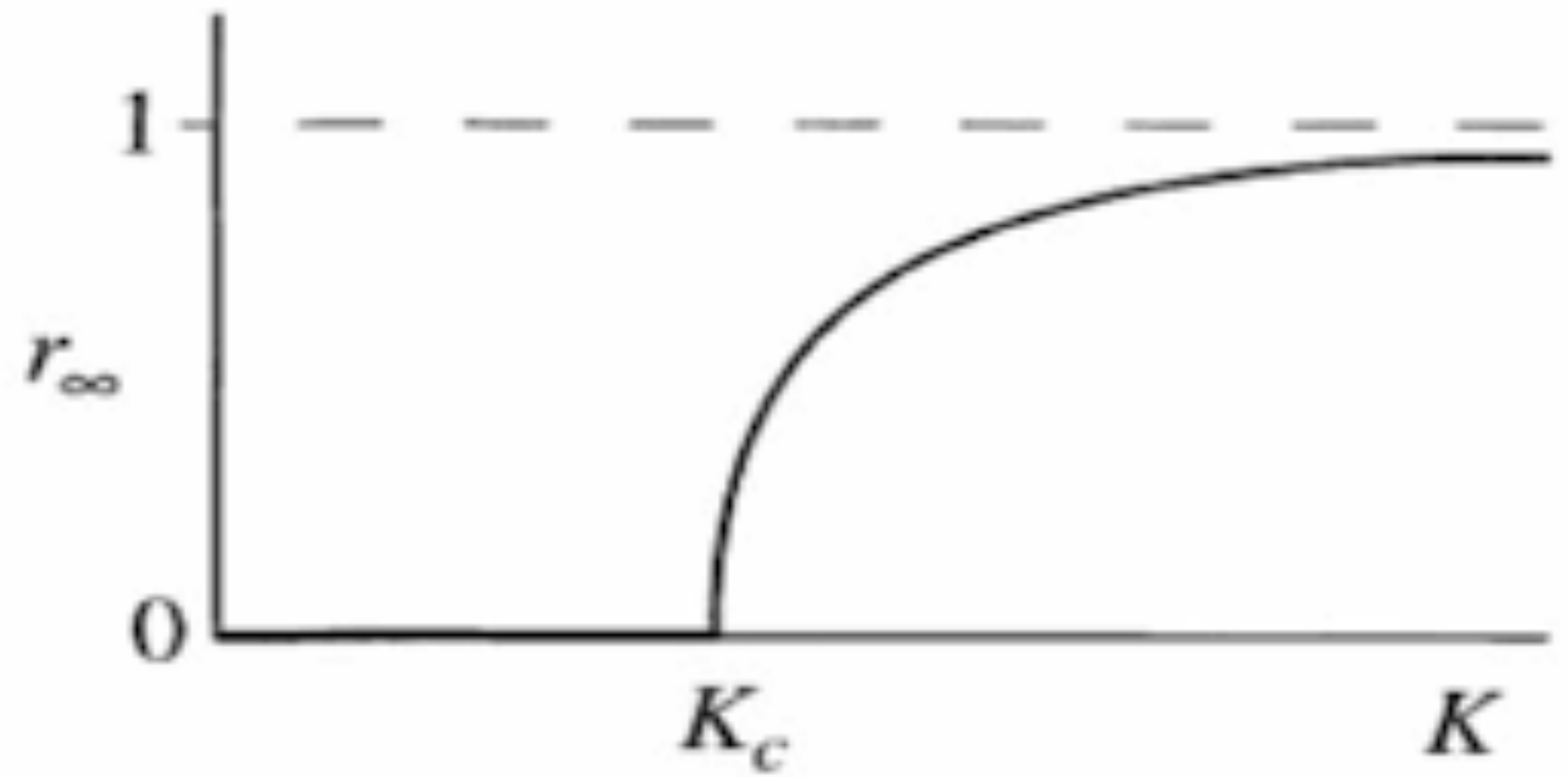
Dynamical regimes



Go play at https://www.complexity-explorables.org/explorables/ride-my-kuramotocycle/

Measure sync: order parameter

$$Z = Re^{i\Phi} = \frac{1}{N}\sum_j e^{i\theta_j}$$



All-to-all: driving by order parameter

$$\dot{\theta}_i = \omega_i + \frac{\gamma}{N}\sum_j \sin(\theta_j - \theta_i)$$

All-to-all: $A_{ij} = 1$

Let's rewrite the second term

$$Re^{i\Phi}e^{-i\theta_i} = \frac{1}{N}\sum_j e^{i\theta_j}e^{-i\theta_i}$$

By multiplying both sides by $e^{-i\theta_i}$

$$\dot{\theta}_i = \omega_i + \gamma R \sin(\Phi - \theta_i)$$

By taking the Imaginary part
And plugging into 1st eq.

Looks like the 2-oscillator equation from before!
Defence on other oscillators j now implicit in R

Each oscillator is driven
by the phase of the order parameter
With a strength proportional to R

Back to group interactions and current research

Multiorder Laplacian

PHYSICAL REVIEW RESEARCH **2**, 033410 (2020)

Multiorder Laplacian for synchronization in higher-order networks

Maxime Lucas,[1,2,3,*] Giulia Cencetti,[4] and Federico Battiston[5,†]

Extended Kumamoto with group interactions

$$
\begin{aligned}
\dot{\theta}_i = \omega &+ \frac{\gamma_1}{\langle K^{(1)}\rangle}\sum_{j=1}^{N} A_{ij}\sin(\theta_j-\theta_i)\\
&+ \frac{\gamma_2}{2!\langle K^{(2)}\rangle}\sum_{j,k=1}^{N} B_{ijk}\sin(\theta_j+\theta_k-2\theta_i)\\
&+ \frac{\gamma_3}{3!\langle K^{(3)}\rangle}\sum_{j,k,l=1}^{N} C_{ijkl}\,\sin(\theta_j+\theta_k+\theta_l-3\theta_i)\\
&+ \cdots\\
&+ \frac{\gamma_D}{D!\langle K^{(D)}\rangle}\sum_{j_1,\dots,j_D=1}^{N} M_{ij_1,\dots,j_D}\sin\left(\sum_{m=1}^{D}\theta_{j_m}-D\,\theta_i\right),
\end{aligned}
\tag{1}
$$

Multiorder Laplacian

PHYSICAL REVIEW RESEARCH **2**, 033410 (2020)

Multiorder Laplacian for synchronization in higher-order networks

Maxime Lucas[1,2,3,*] Giulia Cencetti[4] and Federico Battiston[5,†]

Linearised around sync

$$\delta\dot{\psi}_i = +\frac{\gamma_1}{\langle K^{(1)}\rangle}\sum_{j=1}^{N} A_{ij}(\delta\psi_j - \delta\psi_i)$$
$$+\frac{\gamma_2}{2!\langle K^{(2)}\rangle}\sum_{j,k=1}^{N} B_{ijk}(\delta\psi_j + \delta\psi_k - 2\delta\psi_i)$$
$$+\frac{\gamma_3}{3!\langle K^{(3)}\rangle}\sum_{j,k,l=1}^{N} C_{ijkl}\,(\delta\psi_j + \delta\psi_k + \delta\psi_l - 3\delta\psi_i)$$
$$+\cdots$$
$$+\frac{\gamma_D}{D!\langle K^{(D)}\rangle}\sum_{j_1,\dots,j_D=1}^{N} M_{ij_1,\dots,j_D}\left(\sum_{m=1}^{D}\delta\psi_{j_m} - D\,\delta\psi_i\right).$$

$$L_{ij}^{(d)} = dK_i^{(d)}\delta_{ij} - A_{ij}^{(d)},$$

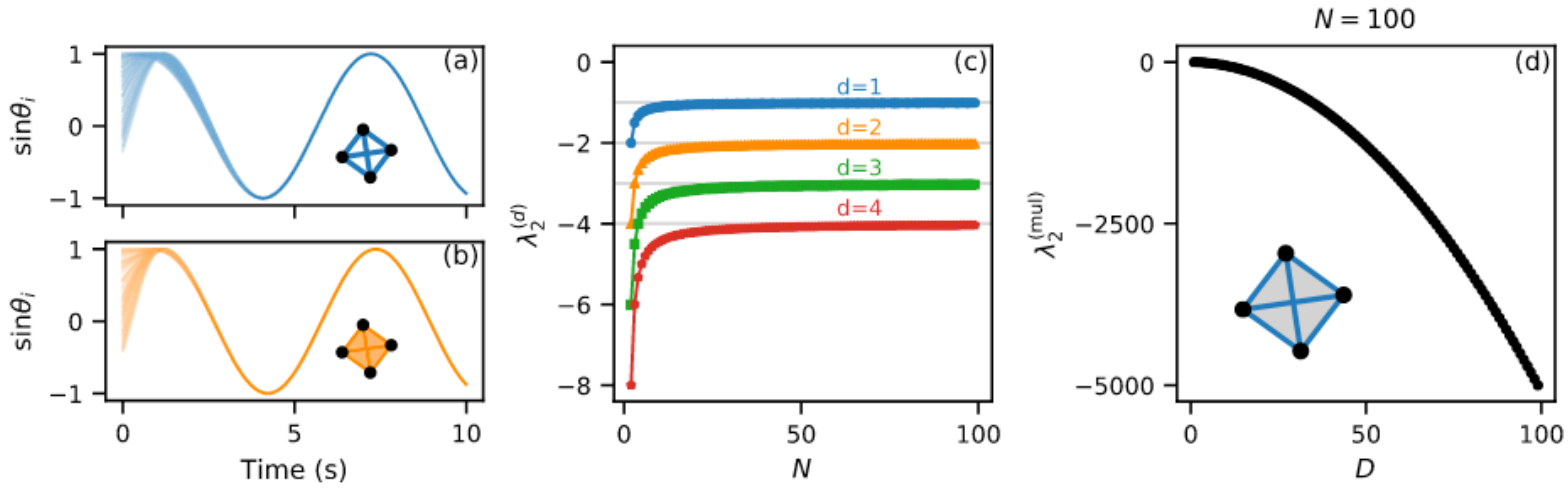
$$K_i^{(d)} = \frac{1}{d!}\sum_{j_1,\dots,j_D=1}^{N} M_{ij_1\dots j_D},$$
$$A_{ij}^{(d)} = \frac{1}{(d-1)!}\sum_{j_2,\dots,j_D=1}^{N} M_{ij_1\dots j_D}.$$

$$\delta\dot{\psi}_i = -\sum_{j=1}^{N} L_{ij}^{(\mathrm{mul})}\delta\psi_j,$$

$$L_{ij}^{(\mathrm{mul})} = \sum_{d=1}^{D}\frac{\gamma_d}{\langle K^{(d)}\rangle}L_{ij}^{(d)},$$

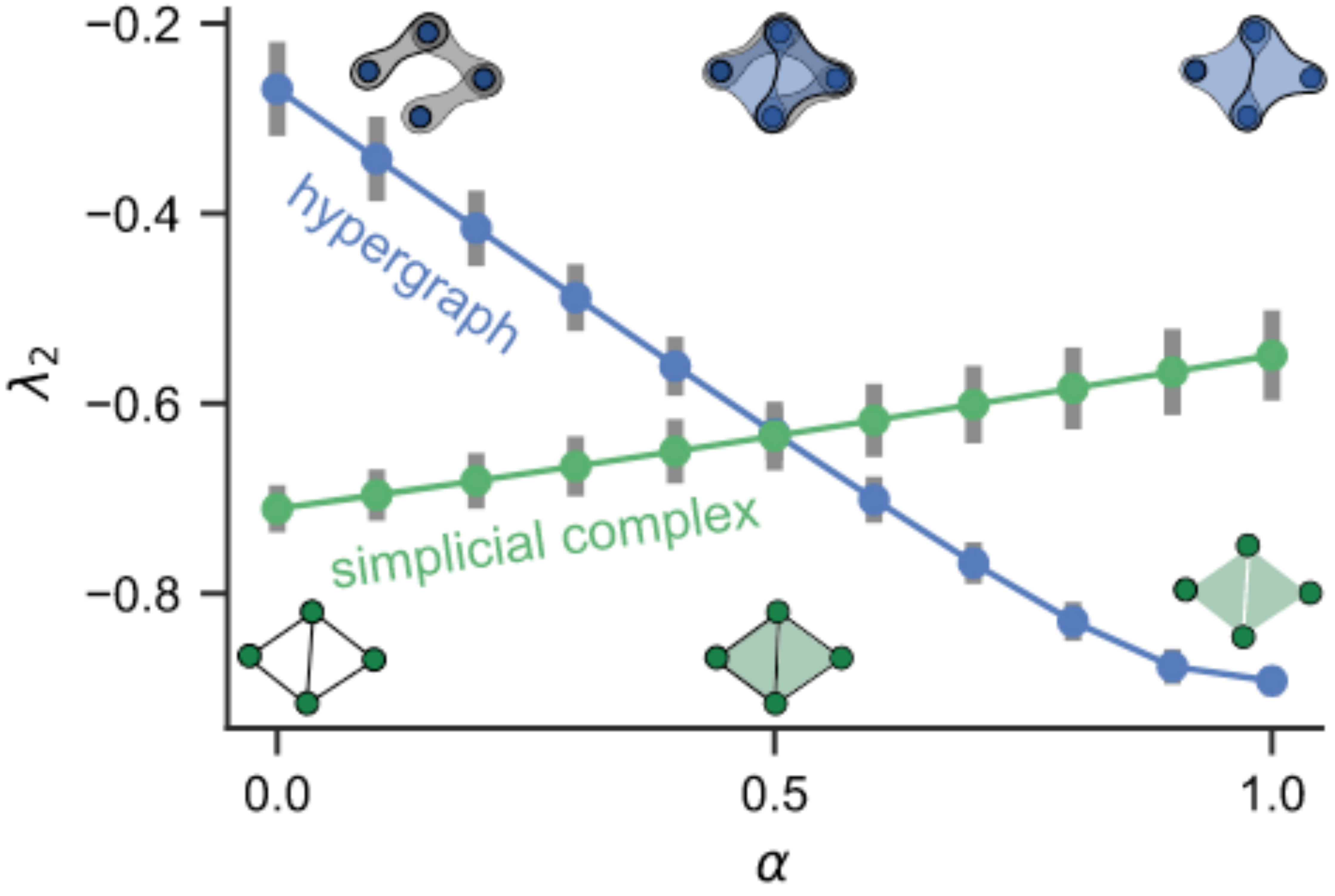
Effect on sync

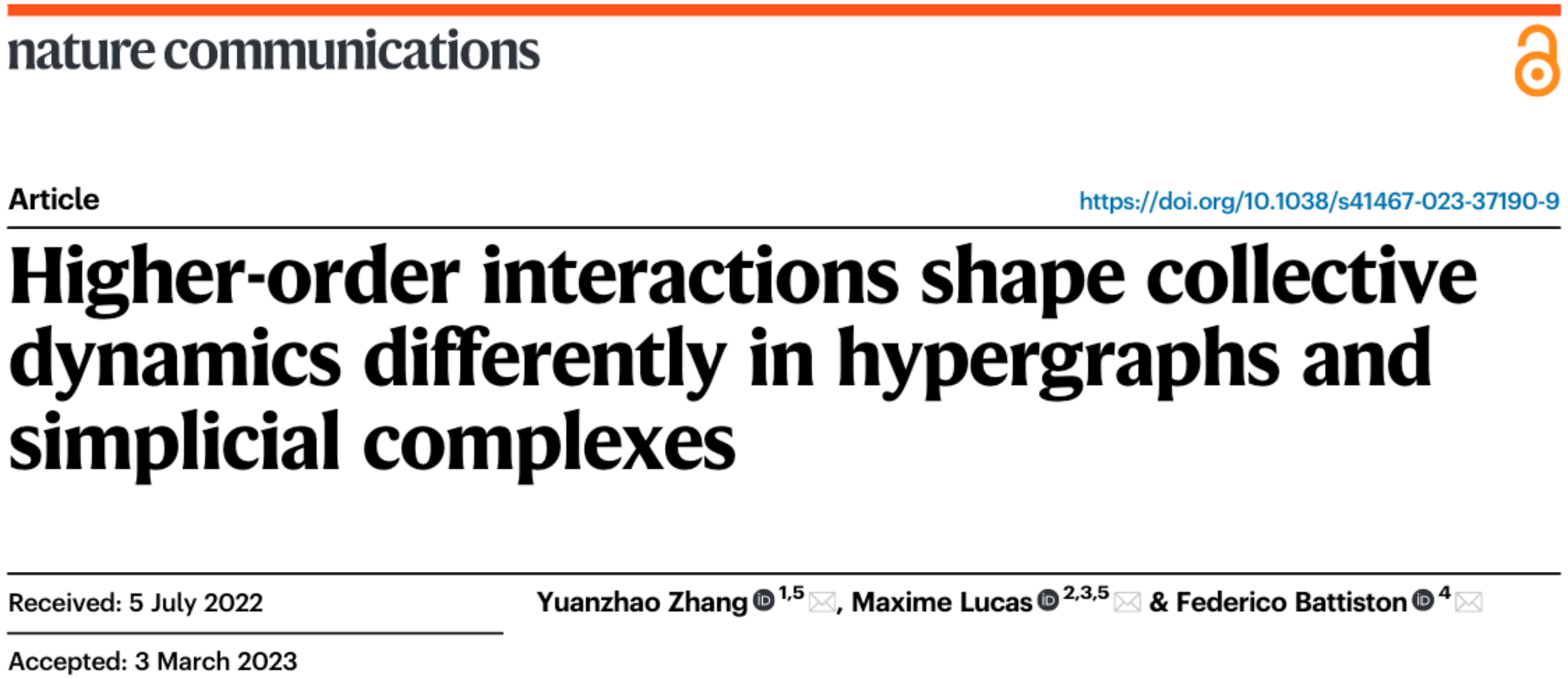
Larger groups sync faster - higher-order stabilise sync



Hypergraphs vs simplicial complexes

They sync differently





nature communications

Article https://doi.org/10.1038/s41467-023-37190-9

Higher-order interactions shape collective dynamics differently in hypergraphs and simplicial complexes

Received: 5 July 2022

Accepted: 3 March 2023

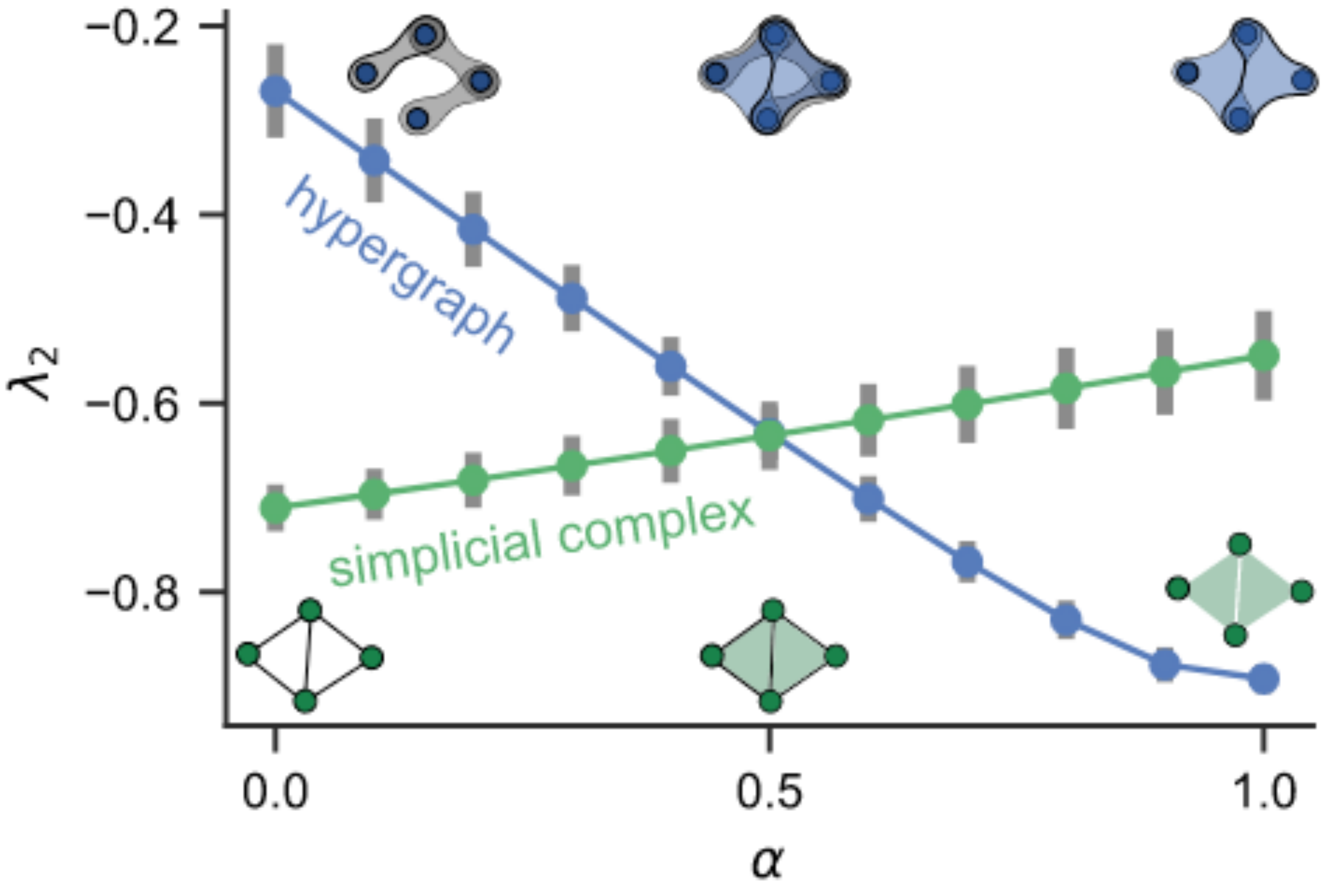
Yuanzhao Zhang [1,5], Maxime Lucas [2,3,5] & Federico Battiston [4]

Always better sync with triangles?

No

$$\dot{\theta}_i = \omega + \frac{\gamma_1}{\langle k^{(1)} \rangle} \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i) + \frac{\gamma_2}{\langle k^{(2)} \rangle} \sum_{j,k=1}^{n} \frac{1}{2} B_{ijk} \frac{1}{2} \sin(\theta_j + \theta_k - 2\theta_i).$$

$$\gamma_1 = 1 - \alpha, \quad \gamma_2 = \alpha, \quad \alpha \in [0, 1].$$



Simplicial Complexes

Rich gets richer

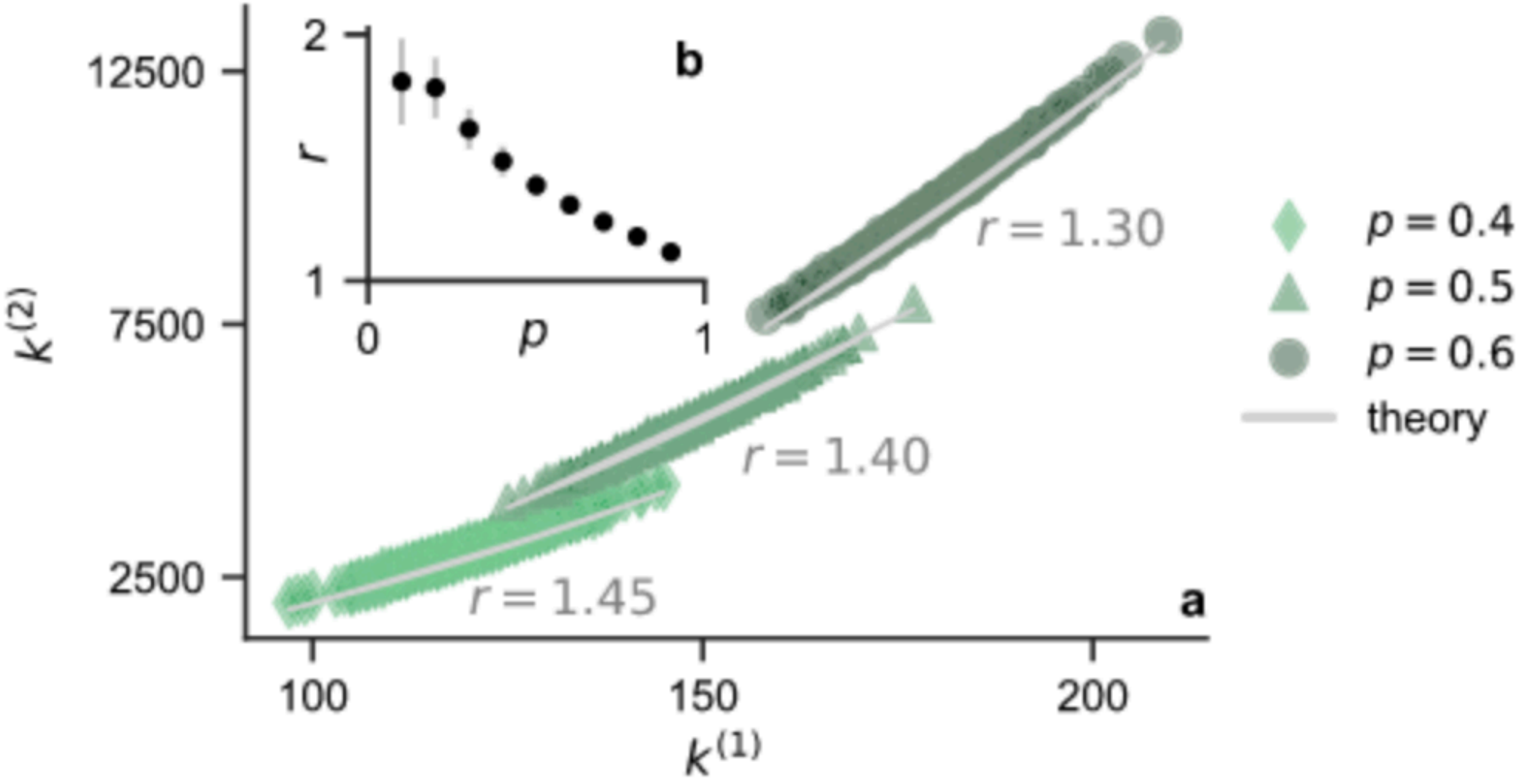
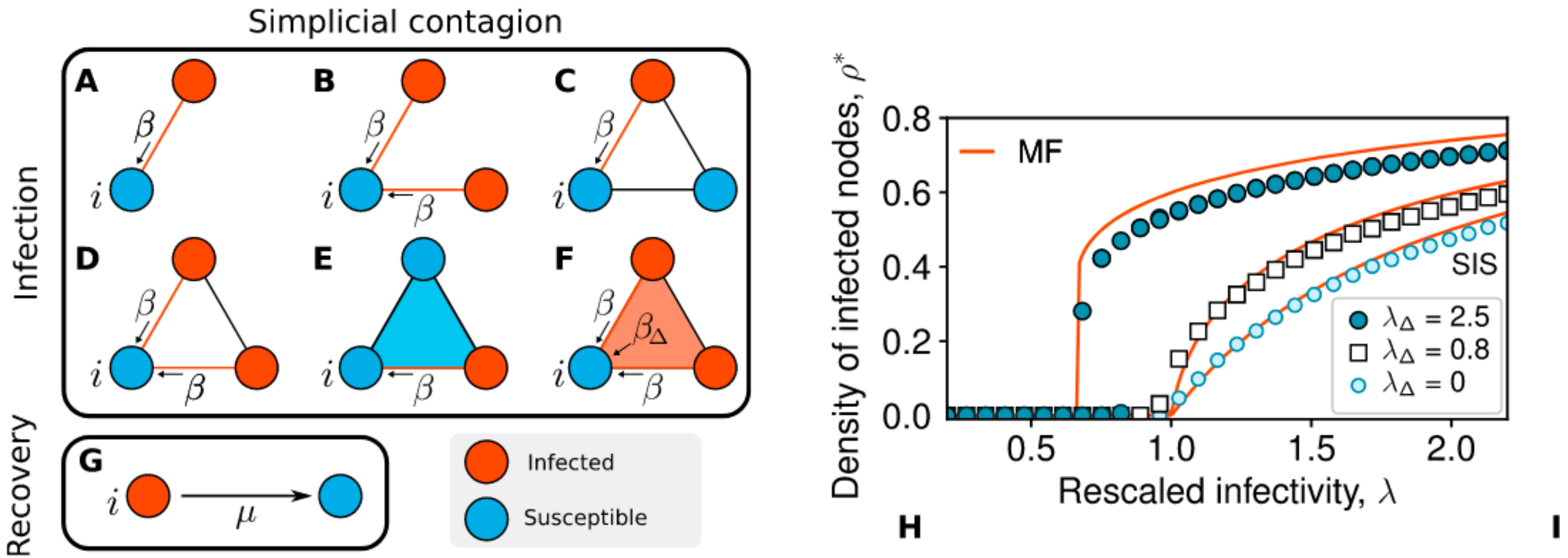


Fig. 3 | Higher order interactions increase degree heterogeneity in simplicial

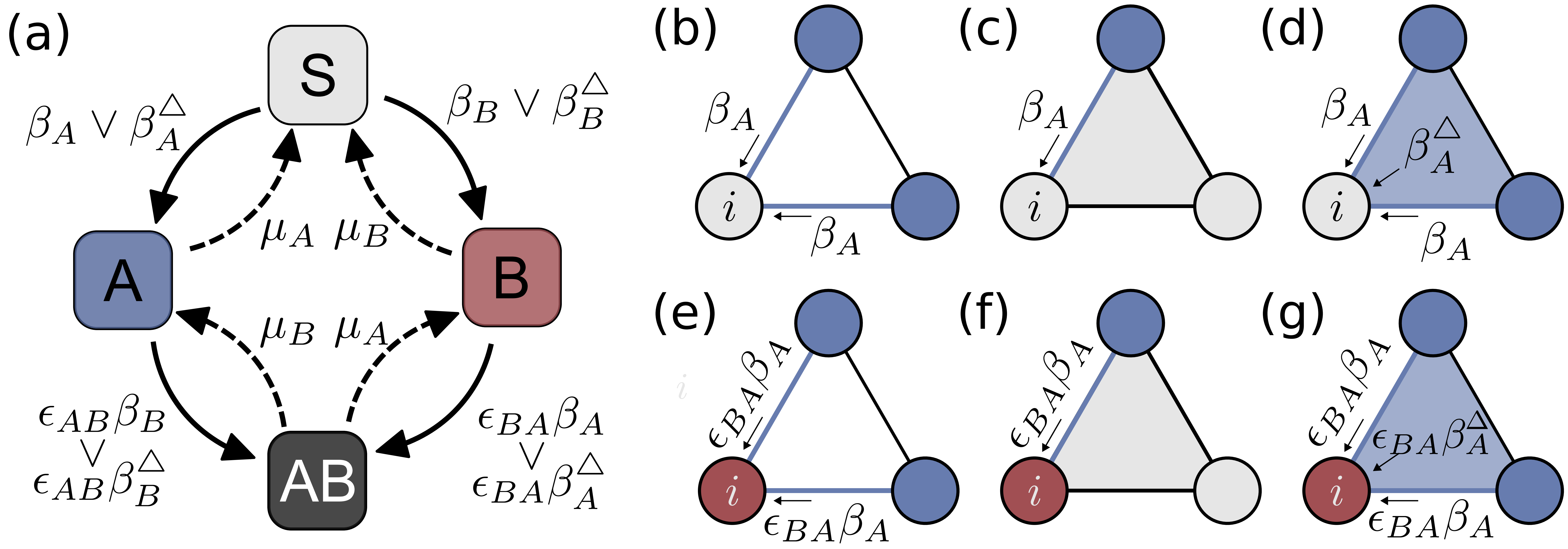
Simplicial contagion

Explosive transition!

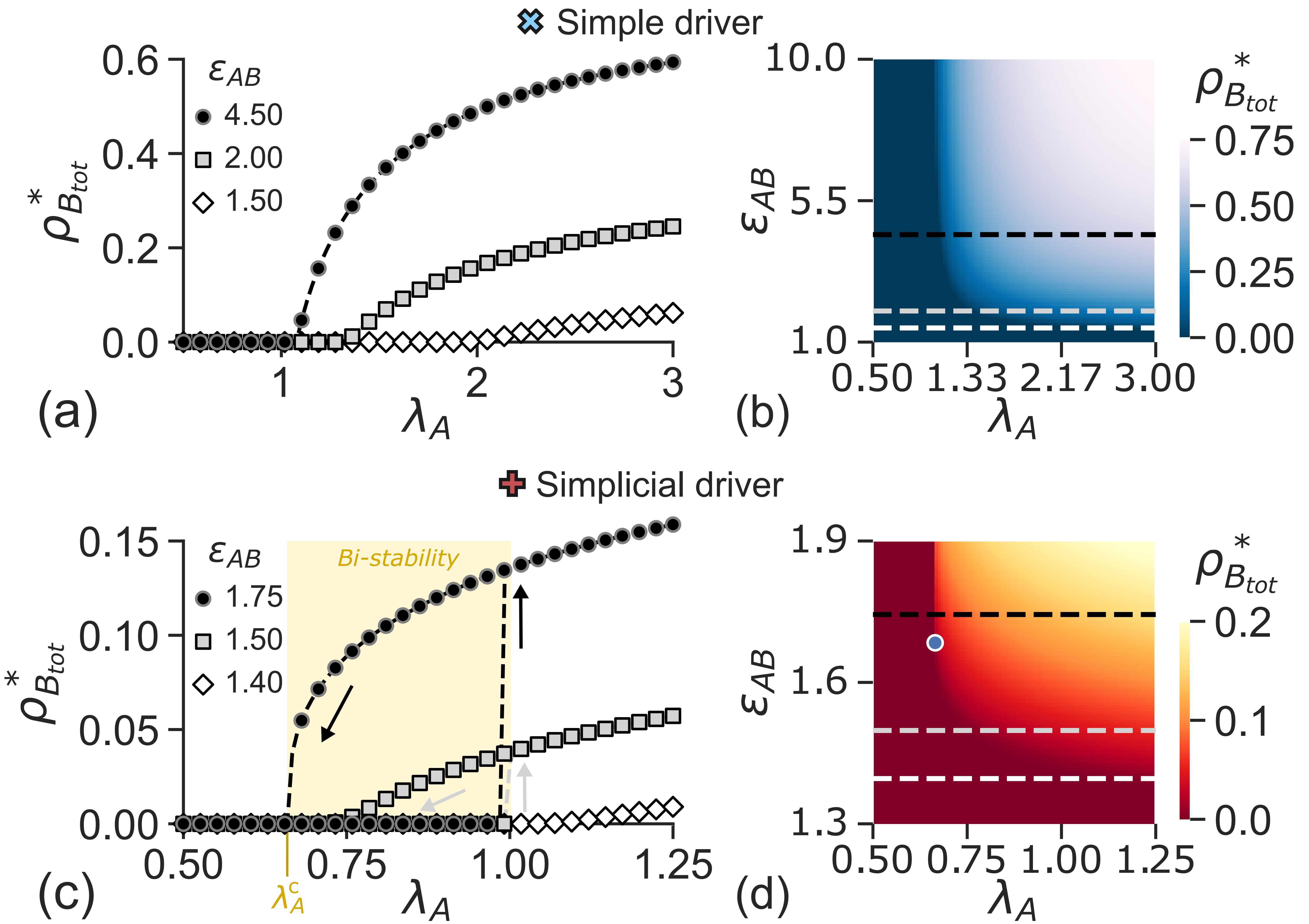


Simplicial driven simple contagion

Unidirectional but also explosive



Simplicial driven simple contagion



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Structure and dynamics: basics

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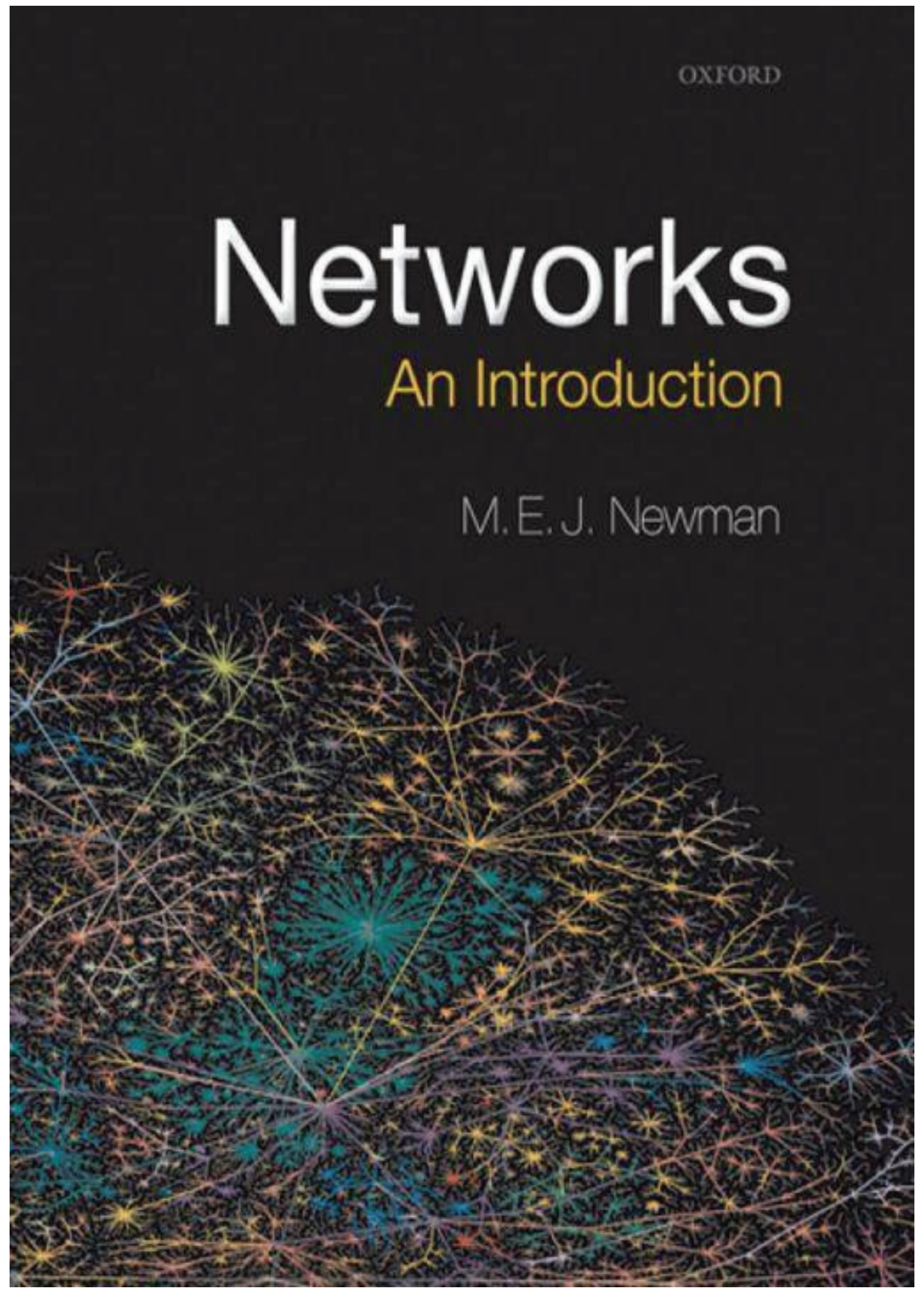
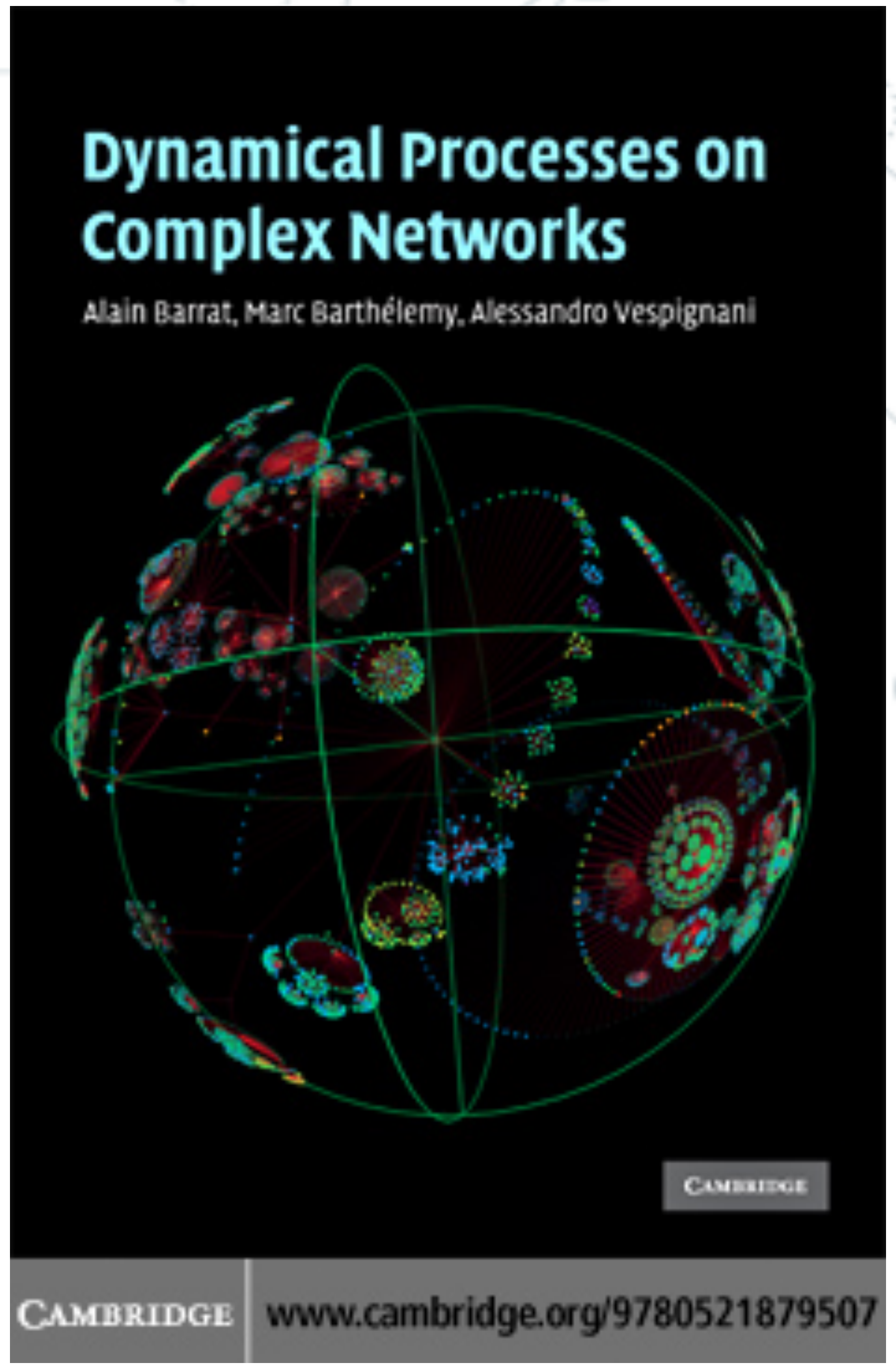
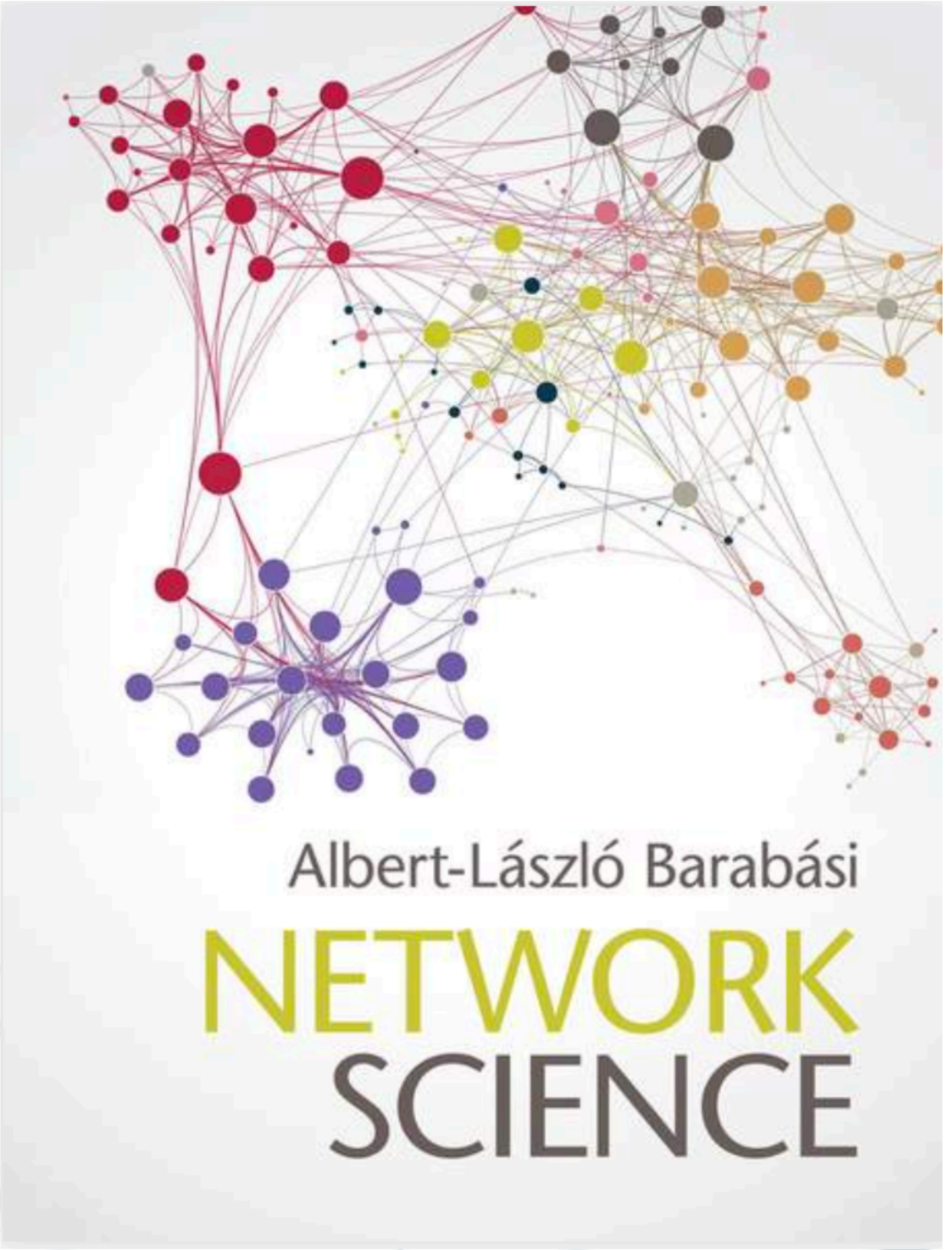
Rev. Mod. Phys. **74**, 47 – Published 30 January 2002







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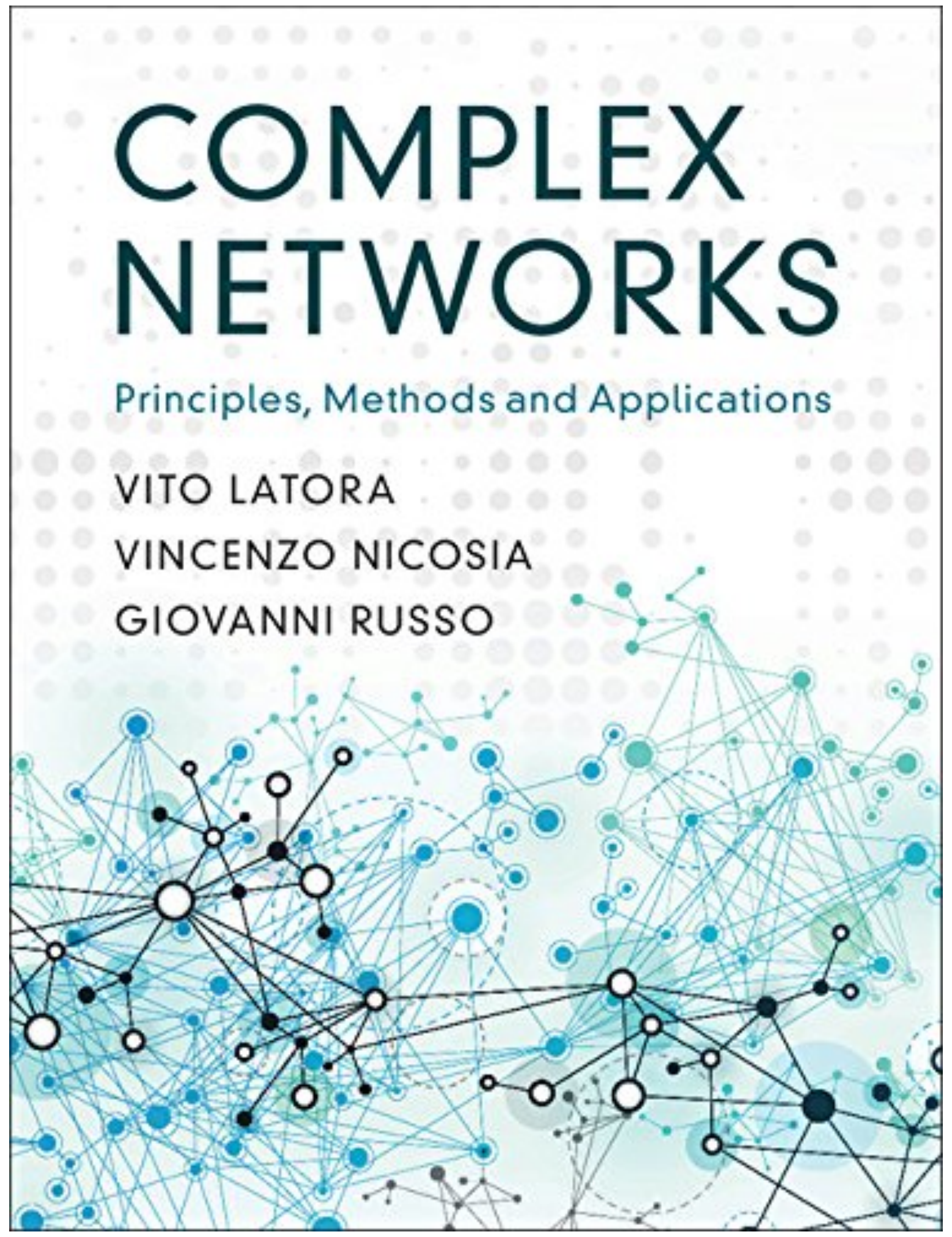

Review materials

Multilayer networks

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Multilayer networks

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S. Boccaletti [a,b,*], G. Bianconi [c], R. Criado [d,e], C.I. del Genio [f,g,h], J. Gómez-Gardeñes [i], M. Romance [d,e], I. Sendiña-Nadal [j,e], Z. Wang [k,l], M. Zanin [m,n]

PHYSICAL REVIEW X 3, 041022 (2013)

Mathematical Formulation of Multilayer Networks

Manlio De Domenico,[1] Albert Solé-Ribalta,[1] Emanuele Cozzo,[2] Mikko Kivelä,[3] Yamir Moreno,[2,4,5] Mason A. Porter,[6] Sergio Gómez,[1] and Alex Arenas[1]

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Alberto Aleta[1,2] and Yamir Moreno[1,2,3]
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Higher-order networks

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https://doi.org/10.1038/s41567-021-01371-4

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DOI 10.1007/s10827-016-0608-6

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SIAM Review
Vol. 63, No. 3, pp. 435–485

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Leo Torres†
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